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AN APOS THEORY ANALYSIS OF THE UNDERSTANDING OF VECTORS INVOLVING PLANE CONCEPTS BY ADVANCED LEVEL MATHEMATICS LEARNERS: A CASE STUDY OF HIGH SCHOOLS IN SHAMVA DISTRICT.

BY

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DECLARATION

I declare that this project hereby submitted by me has not been previously submitted at this or any other University and this is my own work.

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This project has been submitted for examination with my approval as the University Supervisor

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DEDICATION

This research is dedicated to all my family and friends who gave me unwavering throughout the course this research project.

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The researcher would like to express gratitude to all his family, friends and colleagues whose continuous support, encouragement and assistance lead to the success of this project. I also would like to express my heartfelt appreciation to my supervisor Dr Mukavhi L. for his great assistance throughout the research process.

May God bless you.

ABSTRACT

This study investigated the understanding of vectors involving plane concepts among Advanced Level Mathematics learners in Shamva District using the APOS (Action, Process, Object, Schema) theoretical framework. The study aims to analyse students' cognitive development in interpreting planes in three-dimensional space, focusing on normal vectors, vector and Cartesian equations of planes, and their interconnections. This study followed a mixed methods approach with a descriptive design., Questionnaires and interviews were used to gather data from Advanced Level Mathematics teachers and36 pupils who were selected using stratified random sampling from 3 High Schools in Shamva District. Data was collected through diagnostic assessments, structured interviews, and classroom observations to determine students' APOS levels and pre-tests and post-tests. Preliminary findings indicate that most learners operate at the Action and Process levels, demonstrating procedural fluency but struggling with conceptual transitions to Object and Schema thinking. The study further explored how interactive and visual-based learning approaches enhance comprehension and facilitate cognitive progression. Based on the findings, pedagogical recommendations were proposed so as to strengthen conceptual understanding and problem-solving skills in vector geometry. The study findings contribute to Mathematics Education domain by providing insights into effective instructional strategies for teaching 3D vector geometry concepts within the Zimbabwean Advanced Level curriculum. Factors such as lack of knowledge by teachers, lack of adequate lesson time to effectively carry out an APOS analysis in teaching vectors of planes, lack of relevant knowledge and skills to effectively teach the concept, poor self-efficacy among learners. Need for capacitation of schools the provision of workshops and APOS analysis training initiatives were some of the recommendations made to improve the degree of minimizing challenges when dealing with vectors of planes.

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CHAPTER 1: THE RESEARCH PROBLEM

1.0 INTRODUCTION

This chapter will be focusing on the introduction, background of the study, delimitations, statement of the problem, research topic, research objectives, scope of the study, hypothesis, significance of the study, definition of key terms and chapter summary.

This project aims to investigate the understanding of vector involving planes among Advanced Level (A-Level) learners in Zimbabwe using the APOS (Action, Process, Object and Schema) theory. The APOS theory, rooted in constructivist learning, provides a framework for teaching mathematical concepts by progressively developing students' cognitive structures as defined by Oktac, et al (2019).

In the context of vector mathematics, planes can be described using a point on the plane and a normal vector perpendicular to the plane, or using three non-collinear points. (Dorst & De Keninck, 2022). So, in vector planes we will be dealing with the vectors on 3-dimensional plane that is with the x, y and z axes.

This study will design, implement and evaluate a series of instructional interventions based on APOS theory to enhance students' comprehension and application of vectors and vector planes.

1.1 BACKGROUND TO THE STUDY

In Zimbabwe, the Advanced Level Mathematics curriculum covers topics such as vectors and planes. These concepts are foundational for further studies in mathematics, engineering, and related fields. The decline of pass rates in Pure mathematics made the research to ask the examiners about the area in which most of the learners are failing and I found that it’s on vectors on planes. The study indicates that students often struggle to grasp the abstract nature of vector operations, particularly those involving planes (ZIMSEC, 2022) does not publicly announce it). The understanding of vectors involving planes is crucial for Advanced Level learners, especially those pursuing STEM (Science, Technology, Engineering, and Mathematics) fields. Vectors and planes are pivotal in higher-level mathematics, particularly in fields like physics, engineering, and computer science. (Machisi, 2023)

From the investigation, it is noted that the learners are failing to answer the questions on vectors involving planes due to various reasons which include that they are not being taught, they do not have prior knowledge of the distance formula, not able to define perpendicular lines using coordinate geometry, dot product, cross product, equations of planes, vector projections and real objects are not being used by the facilitators (Khan academy, 2024). Also, vectors are being taught once at the end of the syllabus where there will be a little time since the learners will be busy preparing for the end of year examinations. Without the prior knowledge of the mentioned aspects, it will become a challenge for the learners to grasp the concept.

The APOS (Action, Process, Object, and Schema) framework, developed by Ed Dubinsky, offers a comprehensive approach to understanding how students learn mathematical concepts. This study aims to utilize the APOS framework to analyze and improve the understanding of vectors involving planes among Zimbabwean Advanced level mathematics learners. (Bayraktar et al, 2019)

1.2 STATEMENT OF THE PROBLEM

The problem which needs to be solved on the poor of vectors involving planes concepts by Zimbabwean advanced level mathematics learners. So, the research is on an APOS Analysis of the understanding of Vectors involving planes concepts by Zimbabwean Advanced level Mathematics learners

Despite the critical role that vector analysis plays in both higher mathematics and its applications in fields such as physics and engineering, many Zimbabwean Advanced Level Mathematics learners struggle to fully grasp the concepts involving vectors and planes. This challenge is evident in students' performance on examinations and their difficulty in solving problems that require a deep understanding of these abstract concepts. The transition from understanding vectors as mere arrows in space to comprehending their interaction with planes, including concepts such as orthogonality, projections, and intersections, is often marked by significant misconceptions and errors. These difficulties suggest that the traditional instructional approaches may not be effectively facilitating the required cognitive development among learners.

The complexity of vector concepts involving planes necessitates a deeper investigation into how students mentally process and internalize these ideas. However, there is a noticeable gap in research focusing on the cognitive processes that Zimbabwean learners undergo when engaging with these topics. Specifically, there has been limited application of theoretical frameworks, such as the APOS (Action, Process, Object, Schema) theory, which can provide insights into the stages of cognitive development that students experience as they learn these concepts. Without a clear understanding of these cognitive processes, it is challenging to develop teaching strategies that address the specific learning needs of students.

Therefore, the problem that this study seeks to address is the lack of understanding of how Zimbabwean Advanced Level Mathematics learners conceptualize vectors involving planes. This research aims to identify the common cognitive obstacles and misconceptions that hinder learners from achieving a robust understanding of these concepts. By applying the APOS framework, the study will analyze the progression of students' understanding and provide insights that can inform the development of more effective instructional practices tailored to the cognitive needs of learners.

1.3 RESEARCH OBJECTIVES

i. To determine the current level of students' understanding of vector concepts involving vector planes among A-Level learners in Zimbabwe.
ii. To design instructional interventions based on the APOS theory to improve understanding of vectors involving planes.
iii. To implement the APOS-based instructional interventions in selected schools.
iv. To evaluate the effectiveness of the APOS-based interventions on learners' understanding and performance in the vector concepts.

1.4 RESEARCH QUESTIONS

i. What is the current level of students' understanding of vectors involving planes among A-Level learners in Zimbabwe?
ii. How can we design the instructions based on the APOS theory be designed to improve understanding of vectors involving planes.
iii. How can the APOS theory be effectively applied to teach vector concepts involving vector planes?
iv. What impact do APOS-based instructional interventions have on learners' understanding and performance in the vector concepts

1.5 DELIMITATIONS OF THE STUDY

The study focuses on Zimbabwean Advanced Level (A-Level) mathematics learners. These learners are typically students in the final two years of secondary education who are preparing for university entrance exams. The focus on this population is significant because A-Level mathematics is a critical stage where students are expected to develop a deep understanding of advanced mathematical concepts, including vectors and planes.

The study is situated within the Zimbabwean educational system, which is characterized by a diverse range of schools with varying resources, teaching methodologies, and student backgrounds. This context provides a unique opportunity to explore how different factors, such as teaching quality, school resources, and socio-economic backgrounds, influence learners' understanding of vectors involving planes. (Ada, T., & Kurtuluş, A. (2016)

The study specifically examines the understanding of vectors involving plane concepts. These concepts are foundational in advanced mathematics and are essential for various applications in physics, engineering, computer graphics, and economics. The study will focus on several key areas within this topic, such as Vector addition and subtraction, Dot product and cross product, Equations of planes, Intersection of lines and planes, Projection of vectors onto planes and Applications of vectors in solving geometric problems

The study also aims to assess learners' understanding of these concepts at various cognitive levels, from basic procedural skills to more advanced conceptual understanding, as described by the APOS theory. The study is grounded in the APOS (Action, Process, Object, Schema) theory of learning, which is a framework for understanding how mathematical knowledge is constructed in learners' minds. Application of APOS: By applying the APOS theory, the study seeks to categorize and analyze the learners' understanding of vectors and planes, identifying which stage(s) they are operating at and what misconceptions or gaps exist in their knowledge. (Arnon et al, 2014)

The study likely employed a mixed-methods approach, combining quantitative and qualitative data collection techniques. This may include surveys, diagnostic tests, and interviews with students and teachers to gather comprehensive data on learners' understanding. (Şahin & Ozturk 2019)

Tools such as written assessments, concept maps, and clinical interviews will be employed to assess learners' conceptual understanding and to determine which APOS stages they are at concerning different vector-related tasks. The study will target a representative sample of A-Level mathematics learners from various schools across Zimbabwe, ensuring a diverse representation of the student population in terms of geography, socio-economic background, and academic performance.

The study is confined geographically to Shamva District in Mashonaland Central province, Zimbabwe, with an emphasis on capturing the schools in the district. This will allow the study to provide a comprehensive picture of the understanding of vectors involving planes across the country's diverse educational landscape. (Sharma, 2017)

The study will also focus on a specific academic year or term within the Zimbabwean educational calendar. This temporal scope ensures that the findings are relevant to the current state of education and can be used to inform immediate teaching strategies and curriculum development efforts.

In conclusion the scope of this study is carefully defined to provide a focused examination of the understanding of vectors involving planes among Zimbabwean Advanced Level mathematics learners. By targeting a specific population, concentrating on key mathematical concepts, employing the APOS framework, and using a robust methodological approach, the study aims to provide valuable insights into the cognitive processes underlying learners' comprehension of advanced mathematical topics. These insights can then inform educational practices, curriculum development, and further research in mathematics education within Zimbabwe and potentially in other similar educational contexts.

1.6 SIGNIFICANCE OF THE STUDY

1.6.1 ENHANCING CONCEPTUAL UNDERSTANDING

The study focuses on analyzing the understanding of vectors involving plane concepts among Zimbabwean Advanced Level mathematics learners using the APOS (Action, Process, Object, and Schema) framework. This is significant because vectors and planes are fundamental concepts in advanced mathematics, essential for topics such as physics, engineering, computer science, and economics. A deeper understanding of these concepts is critical for students as it forms the foundation for more advanced studies and professional applications. By using the APOS framework, the study aims to identify gaps in understanding and misconceptions, providing valuable insights into the learners' cognitive processes when dealing with vector and plane problems. (Borji et al, 2020)

1.6.2 IMPROVING TEACHING AND LEARNING STRATEGIES

The study's findings can inform educators about the current state of learners' comprehension and the specific areas where they struggle. This is crucial for developing effective teaching strategies tailored to the needs of the learners. Understanding the APOS stages that learners are at can help educators design activities that promote the transition from one stage to another, such as moving from an "Action" understanding (where learners can perform procedures) to a "Process" understanding (where they internalize and manipulate those procedures mentally). This progression is essential for achieving a more comprehensive and flexible understanding of mathematical concepts. (Salgado & Trigueros, 2015)

1.6.3 CURRICULUM DEVELOPMENT

The results of this study can contribute to curriculum development in Zimbabwe and other similar educational contexts. If the study finds, for instance, that most learners struggle with certain aspects of vector and plane concepts, curriculum developers might consider revising the curriculum to introduce these concepts earlier or to provide more detailed explanations and examples. Moreover, it could lead to the integration of specific pedagogical approaches aligned with the APOS framework to enhance understanding. (Mutambara, 2018)

1.6.4 BASIS FOR FURTHER RESEARCH

The study provides a baseline for further research in mathematics education, particularly in the area of vectors and planes. It opens up opportunities for researchers to explore the effectiveness of different teaching interventions designed based on the findings of this study. Additionally, it could encourage similar studies in different regions or educational systems, facilitating a comparative analysis that could enhance global understanding of how different contexts affect the learning of complex mathematical concepts. (Arnon et al, 2014)

1.6.5 ADDRESSING EDUCATIONAL INEQUITIES

The study is significant in addressing educational inequities in Zimbabwe. By identifying specific challenges faced by learners in understanding complex mathematical concepts, targeted interventions can be developed to support underperforming schools or students. This is particularly important in contexts where resources and quality of education can vary significantly. By enhancing understanding and performance in mathematics, the study indirectly contributes to better educational outcomes and future opportunities for students from diverse backgrounds.

1.6.6 CONTRIBUTION TO EDUCATIONAL PSYCHOLOGY

The study contributes to the field of educational psychology by applying the APOS theory to a specific content area (vectors involving planes). This application can help validate and refine the APOS theory in the context of mathematics education. Understanding how students transition from concrete actions to abstract reasoning (schema) in this context can provide insights that are broadly applicable to other areas of cognitive development and learning. (Borji et al, 2018)

1.6.7 ENHANCING LEARNERS' PROBLEM-SOLVING SKILLS

The ability to understand and manipulate vectors and planes is closely tied to problem-solving skills in mathematics. By analyzing learners' understanding through the APOS framework, the study helps in identifying strategies that can enhance these skills. Improving learners' abilities to apply mathematical concepts to solve real-world problems is essential for developing critical thinking and analytical skills, which are highly valued in many fields. (Kazunga & Bansilal, 2020)

1.7 HYPOTHESIS

Zimbabwean Advanced Level Mathematics learners have varying levels of understanding of vector concepts involving planes, which can be analyzed and categorized using the APOS (Action, Process, Object, Schema) theoretical framework.

This hypothesis suggests that the study aims to explore and categorize the different levels of conceptual understanding that students have about vectors in relation to planes, using the APOS framework to identify patterns in their cognitive development. The hypothesis would likely be tested by collecting data from students through assessments, interviews, or observations and analyzing this data within the APOS framework to determine how well students understand the concepts and at which stage of the framework they predominantly operate. (Brijlall, 2020)

1.8 LIMITATIONS OF THE SUDY

1.8.1 SAMPLE SIZE AND REPRESENTATIVENESS

The study may have a limited sample size, which can affect the generalizability of the findings. If the study only involves a small group of Zimbabwean Advanced Level Mathematics learners, the results might not be applicable to a broader population of students in different regions or countries.

1.8.2 CONTEXT-SPECIFIC FINDINGS

The study focuses on Zimbabwean students, which means that the findings are highly contextual and may not apply to students in different educational systems or cultural settings. Different curricula, teaching methods, and cultural factors can affect how students understand mathematical concepts.

1.8.3 METHODOLOGICAL CONSTRAINTS

The study uses APOS (Action, Process, Object, Schema) theory to analyze students' understanding. While this framework is useful for understanding cognitive development in mathematics, it might not capture all aspects of students' learning experiences, such as affective factors (motivation, anxiety) or socio-cultural influences.

1.8.4 RELIANCE ON SELF-REPORTED DATA

If the study includes surveys or interviews, there could be a reliance on self-reported data, which may introduce bias. Students might not accurately report their understanding or difficulties due to social desirability bias or a lack of self-awareness about their cognitive processes.

1.8.5 LIMITED SCOPE OF CONCEPTUAL UNDERSTANDING

The study focuses specifically on vectors involving plane concepts. This narrow focus might overlook other important mathematical concepts that could influence students' overall understanding of vectors and their applications. The study’s findings might not account for the interconnectedness of different mathematical concepts.

1.8.6 POSSIBLE RESEARCHER BIAS

The interpretation of data through APOS theory may involve some degree of researcher bias. The researchers' perspectives and assumptions can influence how they interpret student responses and conceptual understanding.

1.8.7 TEMPORAL LIMITATIONS

The study might capture a specific moment in time, and the findings might not reflect changes in students’ understanding or the impact of different teaching interventions over time. The understanding of concepts like vectors and planes could evolve as students gain more experience and exposure to different mathematical problems.

1.9 DEFINITION OF KEY TERMS

1.9.1 APOS THEORY

APOS (Action-Process-Object-Schema) theory is a framework used in mathematics education to analyze and understand how students learn mathematical concepts. (Oktaç et al, 2019)

1.9.2 VECTORS

In mathematics, a vector is a quantity that has both magnitude and direction. In the context of advanced mathematics, vectors can be represented in two or three-dimensional space, and they are often used to model quantities such as force, velocity, or displacement. Vectors can be added together, scaled by numbers (scalars), and have operations such as dot product (inner product) and cross product defined on them. (Deisenroth et al, 2020)

1.9.3 PLANES

A plane is a flat, two-dimensional surface that extends infinitely in all directions. In the context of vector mathematics, planes can be described using a point on the plane and a normal vector perpendicular to the plane, or using three non-collinear points. (Dorst & De Keninck, 2022).

1.9.4 CONCEPTUAL UNDERSTANDING

Conceptual understanding refers to a deep comprehension of mathematical concepts, operations, and relations. In the context of this research, it pertains to how well students understand the properties and operations involving vectors and planes. (Ermawati & Amalia, 2023).

1.9.5 ADVANCED LEVEL MATHEMATICS LEARNERS

This term refers to students who are studying mathematics at an advanced level, typically in the last two years of secondary education. In Zimbabwe, this corresponds to the A-level (Advanced Level) curriculum, where students engage in higher-level mathematical concepts that prepare them for tertiary education or technical careers. These students engage with complex mathematical concepts including vectors of planes. (Winstone et al, 2022)

1.9.6 MISCONCEPTIONS

Incorrect or incomplete understandings that students develop regarding a particular concept. This study aims to identify common misconceptions that A level Zimbabwean learners have about vectors involving planes. (Chikudo, 2024)

1.10 CHAPTER SUMMARY

The chapter introduced the topic under research and highlighted the possible causes of poor performance in dealing with vectors involving planes. Major assumptions of this study were given as well as questions upon which the research will progress were outlined thereby giving specific objectives of the study. Some important key terms were defined according to how they will be used in the research. The next chapter will dwell on literature review.

CHAPTER TWO: LITERATURE REVIEW

2.1 INTRODUCTION

This chapter provides a comprehensive review of the literature related to the study, focusing on three key areas: the conceptual understanding of vectors involving planes, the application of APOS (Action, Process, Object, Schema) theory in mathematics education, and the challenges faced by Advanced Level mathematics learners in Zimbabwe. This review synthesizes existing research to contextualize the study, identify gaps, and establish a foundation for the investigation of how Zimbabwean Advanced Level learners understand vector concepts in relation to planes.

2.2 UNDERSTANDING OF VECTORS INVOLVING PLANES

Vectors are fundamental in many areas of mathematics and physics, particularly in understanding concepts involving planes. A vector in a plane is a directed line segment with both magnitude and direction, and its applications range from simple geometrical interpretations to complex physical phenomena (Stewart, 2015). The understanding of vectors involving planes typically requires learners to grasp several interrelated concepts, such as vector addition, scalar multiplication, and the dot and cross products (Anton, 2010).

Research suggests that students often struggle with the abstract nature of vectors and the multiple representations used to describe them, such as algebraic, geometric, and physical representations (Harel & Sowder, 1998). These difficulties are exacerbated when vectors are studied in conjunction with planes, where learners must understand concepts such as the vector equation of a plane, normal vectors, and the relationships between different vectors within the same plane (Stewart, 2015).

Learners often find it difficult to visualize three-dimensional planes and vectors, which complicates their ability to solve problems involving intersections, angles, and projections (Mulligan & Watson, 2016).

In addition, Advanced level students frequently struggle with the abstract nature of vector operations, such as the cross product and dot product, leading to errors in determining the relationships between vectors and planes (Davies & Brown, 2017).

The correct application of vector theorems, such as those used to find the equation of a plane or to determine perpendicularity, is often challenging, resulting in frequent misapplications (Lopez & Green, 2020).

Sharma & Patel, 2021 argue that “The complexity of problems that require multiple steps and the integration of various concepts often overwhelms students, particularly when they must combine geometric understanding with algebraic manipulation.”

2.3 APOS THEORY IN MATHEMATICS EDUCATION

APOS theory, developed by Dubinsky and his colleagues, is a framework for understanding how students learn mathematical concepts. The theory posits that mathematical knowledge is constructed through four stages: Action, Process, Object, and Schema (Dubinsky & McDonald, 2001). In the context of vectors and planes, APOS theory can be used to analyze how learners develop their understanding of these concepts.

- **Action:** At the action stage, learners engage in specific tasks involving vectors, such as performing vector addition or finding the dot product, often following a set of instructions without fully understanding the underlying concepts.
- **Process:** As learners repeat these actions and begin to internalize them, they enter the process stage, where they start to see the tasks as part of a broader process. For example, they may begin to understand vector addition as a process of combining magnitudes and directions.
- **Object:** The object stage is reached when learners can conceptualize a process as a complete entity. Here, a learner might recognize a vector not just as an arrow or a list of numbers, but as an abstract object that can be manipulated in various ways.
- **Schema:** Finally, at the schema stage, learners develop a coherent framework that integrates their understanding of vectors and planes with other mathematical concepts, allowing them to solve complex problems and apply their knowledge in different contexts (Dubinsky & McDonald, 2001).

APOS theory has been widely applied in research on mathematics education, particularly in understanding how students develop their mathematical thinking and problem-solving skills. In the context of this study, APOS theory provides a valuable framework for analyzing the cognitive processes that Zimbabwean Advanced Level learners undergo as they learn about vectors involving planes.

2.4 CHALLENGES FACED BY ZIMBABWEAN ADVANCED LEVEL MATHEMATICS LEARNERS

Zimbabwean Advanced Level mathematics learners face a unique set of challenges that can affect their understanding of vectors involving planes. These challenges stem from a variety of factors, including curriculum design, teaching methodologies, and resource availability (Mavhunga & Rollnick, 2013).

2.4.1 CURRICULUM DESIGN: The Zimbabwean mathematics curriculum is often described as content-heavy, with a strong emphasis on rote learning and procedural knowledge (Chikodzi & Mutambara, 2013). This approach can hinder students' ability to develop a deep understanding of vector concepts, as they may focus on memorizing procedures rather than grasping the underlying principles.

2.4.2 TEACHING METHODOLOGIES: Traditional teaching methods, which are prevalent in Zimbabwean classrooms, often involve teacher-centered instruction and limited opportunities for students to engage in exploratory or inquiry-based learning (Mpofu & Mareva, 2013). These methods can restrict students' ability to construct their own understanding of mathematical concepts, such as vectors and planes.

2.4.3 RESOURCE AVAILABILITY: Limited access to learning resources, such as textbooks, technology, and manipulatives, can also impact students' understanding of vectors (Mtetwa & Garofalo, 1989). Without adequate resources, students may struggle to visualize and interact with vector concepts in a meaningful way. Vectors should be taught using real objects and ICT tools which can help the learners to quickly grasp the concept of vectors involving planes

2.4.4 THE IMPACT OF CLASS SIZE ON TEACHING A-LEVEL VECTORS INVOLVING PLANES IN ZIMBABWE

Class size plays a significant role in determining the effectiveness of teaching complex mathematical concepts, such as A-level vectors involving planes. In Zimbabwe, the educational system faces challenges that can exacerbate the difficulties in teaching these advanced topics, particularly when class sizes are large. This essay explores the impact of class size on teaching A-level vectors involving planes in Zimbabwe, with a focus on how it affects student comprehension, teacher-student interaction, and overall educational outcomes.

2.4.4.1 UNDERSTANDING THE CONCEPT: VECTORS AND PLANES

Vectors involving planes are a crucial part of the A-level mathematics curriculum. These concepts require students to understand three-dimensional geometry, vector equations, and the relationships between different geometric entities. Mastering these concepts is not only essential for mathematics but also for subjects like physics and engineering. However, the complexity of these topics can be daunting for students, particularly when they are taught in large classes where individual attention is limited.

2.4.4.2 CLASS SIZE AND STUDENT COMPREHENSION

One of the most significant impacts of large class sizes is the reduction in student comprehension. Research has shown that smaller class sizes are associated with better academic performance, particularly in subjects requiring higher-order thinking skills (Blatchford et al., 2020). In large classes, students may struggle to grasp complex concepts like vectors and planes because teachers are unable to provide the personalized attention necessary to address individual learning gaps. This is particularly problematic in Zimbabwe, where the average class size in secondary schools can exceed 40 students, making it challenging for teachers to ensure that all students are keeping up with the material.

2.4.4.3 TEACHER-STUDENT INTERACTION

Effective teaching of A-level vectors requires significant teacher-student interaction. In smaller classes, teachers can engage with students more frequently, provide detailed explanations, and offer immediate feedback. However, in larger classes, the teacher's attention is spread thin, making it difficult to identify and assist struggling students. According to a study by Muchemwa and Madondo (2021), in Zimbabwe, large class sizes often result in teachers adopting a lecture-based approach rather than interactive teaching methods, which are more effective for complex topics like vectors. This shift in teaching style can further hinder student understanding and retention of the material.

2.4.4.4 IMPACT ON EDUCATIONAL OUTCOMES

The impact of class size on educational outcomes is evident when comparing student performance in schools with varying class sizes. Studies have shown that students in smaller classes perform better in mathematics and science subjects, including topics like vectors involving planes (Muchemwa, 2022). In Zimbabwe, where resources are limited, and class sizes are often large, students may not receive the support they need to excel in these subjects. This can have long-term implications for their academic and professional futures, particularly in fields that require strong mathematical skills.

2.4.4.5 ADDRESSING THE CHALLENGES

To mitigate the negative effects of large class sizes, several strategies can be employed. Schools can adopt a more differentiated teaching approach, where students are grouped according to their ability levels, allowing teachers to tailor their instruction to the needs of each group. Additionally, the use of technology, such as online tutorials and interactive simulations, can help reinforce concepts outside the classroom, providing students with additional opportunities to practice and understand the material.

Furthermore, investment in teacher training is crucial. Teachers need to be equipped with strategies for managing large classes effectively and for using available resources to enhance student understanding. The Zimbabwean government, in collaboration with educational stakeholders, should prioritize reducing class sizes or implementing support systems to ensure that students receive the attention and instruction necessary to succeed in challenging subjects like A-level vectors.

In conclusion Class size significantly impacts the teaching and learning of A-level vectors involving planes in Zimbabwe. Large classes hinder student comprehension, reduce teacher-student interaction, and negatively affect educational outcomes. Addressing these challenges requires a multifaceted approach, including smaller class sizes, differentiated instruction, the use of technology, and improved teacher training. By implementing these strategies, Zimbabwe can enhance the quality of education and better prepare its students for the demands of higher education and the workforce.

2.5 SYNTHESIS OF THE LITERATURE

The literature reviewed in this chapter underscores the complexity of learning vectors involving planes and the importance of understanding the cognitive processes involved. APOS theory offers a robust framework for analyzing these processes, and its application in this study will provide insights into the specific challenges faced by Zimbabwean learners. The review also highlights the need for educational reforms in Zimbabwe to address the challenges posed by the current curriculum, teaching methodologies, and resource limitations.

2.6 CONCLUSION

This chapter has provided a detailed review of the literature related to the understanding of vectors involving planes, the application of APOS theory in mathematics education, and the challenges faced by Zimbabwean Advanced Level mathematics learners. The insights gained from this review will inform the subsequent chapters of this dissertation, particularly the methodology and analysis sections, as the study seeks to explore how APOS theory can be used to improve the understanding of vector concepts among Zimbabwean learners.

CHAPTER 3: RESEARCH METHODOLOGY

3.0 INTRODUCTION

In this section, the researcher will focus on the plan of gathering data and the procedures for conducting the research. The research paradigm, the research design, research methods, reliability and validity, ethical issues, sampling procedure and research instruments will be deliberated on with the benefits and limitations of using each being spelt out. The data collecting plan and data analysis plan will be discussed

This chapter provides a detailed description of the methodology employed to examine the understanding of vectors involving plane concepts among Zimbabwean Advanced Level Mathematics learners, using the APOS (Action, Process, Object, and Schema) theoretical framework. The focus of this chapter is on explaining the chosen research design and its justification, identifying the target population, detailing the sampling procedure, outlining the instruments used to gather data, describing the data collection procedures, and presenting the data analysis plans. Each section of this chapter justifies the choices made in terms of their relevance to the research objectives and their appropriateness for investigating learners' conceptual understanding of vectors and planes.

3.1 RESEARCH DESIGN

A research design is a plan of the study which provides a framework for collecting data. The design of the research is a plan for data collection. We can minimize the effects of error variance by choosing a design that permits us to calculate the contribution of one or more nuisance variables (Myers et al 2013). In other words, the design indicates how the research is set up, what happens to participants and what methods of data collection will be used. The research design will enable the researcher to anticipate what appropriate research decisions should be made to maximise validity of the results.

This study adopted a descriptive survey design, which is suited to understanding how a large group of learners conceptualizes mathematical concepts like vectors involving planes. A descriptive survey was chosen because it facilitates the collection of both quantitative and qualitative data, which is crucial for the application of APOS theory in assessing the stages learners pass through in their understanding of mathematical concepts (Creswell, 2014). The design's strength lies in its flexibility, allowing for a broad exploration of the research problem without manipulating the learning environment. This non-experimental approach is ideal for educational research where natural behavior is observed, and responses are recorded based on existing conditions (Bryman, 2015).

The descriptive survey design was selected because it enables the researcher to collect detailed data from a wide cross-section of students, reflecting their varying levels of understanding of the mathematical concepts involved (Mertens, 2015). Additionally, this design supports the analysis of both cognitive and procedural elements of the students' responses, as necessitated by the APOS framework, which emphasizes how learners move from performing actions on mathematical objects to constructing internal processes, and finally organizing these into coherent schemas (Dubinsky & McDonald, 2018).

3.2 TARGET POPULATION

The target population for this study comprised Advanced Level Mathematics learners in Shamva District, Mashonaland Central in Zimbabwe, particularly those studying vectors involving planes in their final year of the A-Level syllabus. The population was selected from a diverse group of secondary schools located in both urban and rural areas, providing a broad socio-economic spectrum that could influence learners' understanding of the concepts. The demographic focus was on learners aged between 17 and 19 years. This age group was chosen as it represents students in the final stages of their secondary education, who are expected to have been exposed to the entire vectors and planes syllabus.

The study covered an estimated 36 students, across 3 schools in Shamva district, Mashonaland central province. These schools were chosen because they provide a diverse educational landscape that includes both well-resourced and under-resourced schools, allowing for a balanced investigation into how learners from different environments conceptualize vectors in planes. The data collection period was set for September to October 2024, during which learners were actively engaged with vector-related topics in the mathematics curriculum (ZIMSEC, 2022).

3.3 RESEARCH METHODS

Research methods can be defined as “a systematic and scientific procedure of data collection, compilation, analysis, interpretation, and implication pertaining to any problem” (Bajpai, 2011). A mixed methods approach was used to cater for both qualitative and quantitative data collected from Advanced Level teachers and pupils.

3.3.1 POPULATION AND SAMPLING

A population in statistics and other areas of mathematics is a discrete group of people, animals or anything that can be identified by at least one common characteristic for the purpose of data collection and analysis (Punch, 2013). In this regard, the population consists of all Advanced level learners doing mathematics. All being Advanced levels and mathematics students, data can be collected for analysis as they have the same characteristics.

Advanced level pupils were chosen to be the participants of this study because much of the content of Vectors is done at Advanced level. Looking at the time data was collected; Advanced levels were the ideal group as they were always available at the school and online since the school is conducting formal online lessons during the holidays.

A sample is a fraction of the total population to be used in each study. It is a representative of the whole population (Taherdoost, 2016). In this study, the sample consists of thirty-six Form Six pupils and four mathematics teachers from three A level High Schools in Shamva District. Since Advanced level pupils at these schools' total to one hundred and twenty, a sample of thirty-six pupils constitute thirty percent of the Advanced level pupils' population will be used. This is more than the expected sample size for a quantitative study, that is, 30 percent of the population is required for a quantitative study. The probability sampling procedures, called systematics random sampling and simple random sampling, will be used so that selection of participants depends purely on chance not on personal bias.

3.3.1.1 STRATIFIED SAMPLING

A stratified random sampling technique was employed to select participants from the target population. The sample consisted of 36, representing 30% of the population. Stratification was used to ensure that various subgroups such as students from rural and urban schools, high-achieving and low-achieving students were adequately represented (Teddlie & Yu, 2007). This method is appropriate because it reduces sampling bias and enhanced the generalizability of the findings by ensuring proportional representation from different school types.

3.3.1.2 SIMPLE RANDOM SAMPLING

Simple random sampling is defined as a way of sampling in which participants of a given populace have the same possibility of being selected. In other words, simple random sampling means that every member of the sample is selected from the group of population in such a manner that the probability of being selected for all members in the study group is the same (Jawale, 2012). To get an ideal random sample, the population should be finite, and the members of the populace should be determined and listed to prevent bias. In this regard, the population consists of one hundred and twenty Advanced level pupils at three High Schools in Shamva District as indicated by class registers.

Within each stratum, simple random sampling was used to select participants, ensuring that every student had an equal chance of being included in the study. This approach was justified by the need to generate a representative sample that mirrors the diversity of the entire population (Cohen, Manion, & Morrison, 2018). A sample size of 36 was considered appropriate to balance the need for sufficient statistical power while maintaining practical manageability during data collection and analysis (Lapan, Quartaroli, & Riemer, 2016).

3.3.1.2.1 ADVANTAGES OF SIMPLE RANDOM SAMPLING

Simple random sampling ensures that every element in the sampling frame has an equal chance of being included in the sample. Therefore, it is free from bias and prejudice. Another advantage is that simple random sampling secures a spread of sample across the entire population. Simple random sampling ensures that every individual has equal probability of being selected, thus ensuring a representative sample. Furthermore, it needs minimum knowledge of the study group of population in advance (Etikan & Bala, 2017).

3.3.1.2.2 DISADVANTAGES OF SIMPLE RANDOM SAMPLING

Simple random sampling is associated with some limitations. White (2015) notes that it can carry large errors from the same sample size than that are found in stratified sampling. This means that random sampling can pick pupils with the same performance. Unlike in stratified sampling where pupils could be grouped according to performance, that is, as best performers, average performers and bad performers. The process of selection can interact with a hidden periodic trait within the population such that the representative from the sample is compromised.

3.3.2 DATA COLLECTION INSTRUMENTS

Data collection instruments are tools used for collecting information and data required to find solutions to the problem under study. A research instrument is defined as a testing device for measuring given phenomena such as pen and paper test, questionnaire, interviews or a set of guidelines for observation. In other words, it refers to the way in which data is going to be collected. In this study, the researcher is going to use a pen and paper test and a questionnaire. These instruments will be used to gather data for the purpose of the study and are deemed necessary since the research is quantitative in nature (Zohrabi, 2013).

The researcher will administer a pre-test to identify the types and source of learner shortcomings in word problems in Vectors involving planes. Marks obtained by pupils in the pre-test will be recorded. The researcher will design activities that will help to address the shortcomings that would have been noted. In the activities pinpoint intervention will be done. Pupil participants will be given questionnaires to respond to after the pre-test. Mathematics teachers will also get their questionnaires. After the administration of the pre-test, responding to the questionnaire and remedy have been given, a post-test will be given and marked. The marks they obtain will be recorded. Questionnaires will be collected, and data grouped according to the responses given by participants. The data will then be analysed.

3.3.2.1 QUESTIONNAIRE

A questionnaire is a list of questions for which answers are being sought. A questionnaire can be defined as a set of printed or written questions with a choice of answers devised for the purpose of a survey or statistical study. For this study, structured questionnaire with closed ended questions will be used. A closed ended questionnaire is the one in which the respondents can fill in answers such as yes or no, agree or disagree. This makes it easy for pupils who are participants to this study to answer quickly and it will simplify the data analysis process. A questionnaire is objective in nature (Taherdoost, 2016).

3.3.2.1.1 ADVANTAGES OF QUESTIONNAIRE

Wilson (2013) asserts that the questionnaire makes it possible to give similar or standardised questions to the respondents. Hunter (2012) notes that many participants can be reached in a short period of time thereby allowing the collection of vast amounts of information. The results can usually be quickly and easily quantified by either a researcher or using a software package. Questionnaires also reduce the chances of evaluation bias as the same questions are asked to all respondents. Also, administration is comparatively inexpensive and easy when gathering data even from a large sample. Data on a questionnaire is easy to represent on different methods of representing the data such as bar charts, pie charts and line graphs. The data will show a continuous distribution. The questionnaire can be answered online using different social media platforms. Questionnaires can be sent, answered and sent back to the researcher.

3.3.2.1.2 DISADVANTAGES OF A QUESTIONNAIRE

Questionnaires have their own limitations. A major setback is that the meaning of the question can be distorted. Patten (2016) argues that people may read differently into each question and therefore respond basing on their own interpretation of the question. Non-structured questionnaires may restrict the respondents to specify answers.

The researcher will, however, ensure that the return rate is high by delivering the questionnaires to pupils for them to respond during lessons time. Pupils will also be told why the information is being collected and how the results will be beneficial so as to erase any suspicion in them. This will make them respond honestly with a positive opinion. The questionnaires will also be anonymous.

3.3.2.2 PRE-TEST AND POST TEST

This is a quick and easy way of assessing pupils. The pen and paper test will come in two forms, the first being a pre-test and the other a post test. A pre-test is the one that is given to participants before they are given awareness. This will be used to identify pupil errors in word problems in geometry. A post-test is the one given to participants to see if change has occurred in the outcome after going through a particular course. This will come as a follow up to see if what the participants had learnt has managed to change their performance for the better (Yasin et al, 2016).

3.3.2.2.1 ADVANTAGES OF PRE AND POST TEST

Pre and post-test analysis has several advantages. The post-test was reviewed with the students after it was administered. The pre/post-tests and the questions on the final exams used for this study were designed to assess student mastery of these learning objectives within the context of each course (Boyas et al, 2012). Assessment can be standardised or locally developed. Administering a test of entry behaviour can determine whether assumed pre-requisites have been achieved. Post-test allows to measure a pupil performance to see if the course has had an effect. As such it is a useful method for measuring the value added by a program of study. Knowledge of the status of a group may provide guidance for future activities as well as the basis for comparison. In other words, the pre and post-test complement each other. The researcher will employ both the pre-test and the post-test to add validity to the results of the study. These tests can also be answered online using social media and other online platforms (Janelli & Lipnevich, 2021).

3.3.2.2.2 DISADVANTAGES OF PRE-TEST AND POST TEST

If using the same test for both pre and post-test, students will absorb knowledge just from taking the test and will attend more readily to the content (Yasin et al, 2016). In this study, the researcher will use different tests to establish the true performance of participants in solving problems on vectors involving planes. In some instances, however, students can complete the pre-test and not the post-test or vice versa; these test results would be eliminated from the study findings (Bryan & Karshmer, 2013).

3.3.2.3 INTERVIEWS

An interview is a conversation for gathering information. A research interview involves an interviewer, who coordinates the process of the conversation and asks questions, and an interviewee, who responds to those questions. Interviews can be conducted face-to-face or over the telephone. The internet is also emerging as a tool for interviewing. In this study the researcher used interviews to collect data from the teachers. Qualitative information is obtained from the teachers through interviewing them. (Nashon, 2021)

3.3.2.3.1 ADVANTAGES OF INTERVIEWS

Interviews are an appropriate method when there is a need to collect in-depth information on people’s opinions, thoughts, experiences, and feelings. Interviews are useful when the topic of inquiry relates to issues that require complex questioning and considerable probing. Face-to-face interviews are suitable when your target population can communicate through face-to-face conversations better than they can communicate through writing or phone conversations (e.g., children, elderly or disabled individuals). (Nol, 2021)

3.3.2.3.2 DISADVANTAGES OF INTERVIEWS

Conducting interview studies can be very costly as well as very time-consuming. An interview can cause biases. For example, the respondent's answers can be affected by his reaction to the interviewer's race, class, age or physical appearance. Interview studies provide less anonymity, which is a big concern for many respondents. There is a lack of accessibility to respondents (unlike conducting mailed questionnaire study) since the respondents can be in around any corner of the world or country. (Nol, 2021)

3.3.3 DATA ANALYSIS METHODS

After data collection, the data will be presented and analysed. Firstly, the marks obtained in the pre-test and post-test will be presented in the form of a table. Each participant will have two recorded marks. Z-test to establish whether the pre-test and the post-test mark will be the same or the marks will be different. The responses from questionnaires will be tabulated and graphs produced. These will be analysed and commented on. The test results of learners when solving problems on vectors involving planes will be tabulated for both the pre and post-test. The Z-test will be used to examine the meaningful relationship between each type of necessary knowledge and all the occurring shortcomings. The findings will be commented on. The averages and standard deviations will be found to carry out a significant test using the Z-test.

3.4 RELIABILITY AND VALIDITY

Reliability refers to the extent to which the same answers can be obtained using the same instruments more than one time (Pallant, 2020). According to Wilson (2015) reliability issues are most of the time closely associated with subjectivity and once a researcher adopts a subjective approach towards the study, then the level of reliability of the work is going to be compromised.

Validity of research can be explained as an extent at which requirements of scientific research method have been followed during the process of generating research findings. Field (2017) considers validity to be a compulsory requirement for all types of studies. There are different forms of research validity and main ones are specified by Cohen et al (2014) as content validity, criterion-related validity, construct validity, internal validity, external validity, concurrent validity and face validity.

Measures to ensure validity of a research include, but not limited to the following points: Appropriate time scale for the study must be selected; Appropriate methodology is to be chosen, considering the characteristics of the study; The most suitable sample method for the study is to be selected; The respondents must not be pressured in any ways to select specific choices among the answer sets.

It is important to understand that although threats to research reliability and validity can never be eliminated, however the researcher will strive to minimize this threat as much as possible.

3.5 ETHICAL CONSIDERATIONS

3.5.1 INFORMED CONSENT

All participants will provide informed consent before participating in the study. This means that participants will be fully informed about the purpose of the study, what their participation will involve, any potential risks or benefits, and their right to withdraw at any time without any negative consequences. For the minors (under 18), consent will also be obtained from their parents or guardians.

3.5.2 CONFIDENTIALITY AND ANONYMITY

The study will ensure that the privacy and confidentiality of all participants are maintained. This includes anonymizing data so that individual participants cannot be identified in any reports or publications arising from the study. Personal information and data will be securely stored and only accessible to authorized personnel to protect the participants' privacy.

3.5.3 MINIMIZATION OF HARM

The research will be designed in such a way that it does not cause any psychological, emotional, or physical harm to the participants. For instance, this study should not put undue stress on students or make them feel uncomfortable or anxious about their performance or understanding of mathematics.

3.5.4 VOLUNTARY PARTICIPATION

Participation in the study will be voluntary, and students will not be coerced or pressured into taking part. They should be free to decline or withdraw from this study at any point without any repercussions, and this will be clearly communicated to them.

3.5.5 RIGHT TO WITHDRAW

Participants will be allowed to withdraw from the study at any stage without providing a reason. This right will be communicated clearly to ensure participants understand that their decision to withdraw will not affect their standing in their educational institution or any future participation in other studies.

3.5.6 AVOIDING CONFLICTS OF INTEREST

The researcher by all means try to avoid any conflicts of interest that could influence the study's design, implementation, or interpretation of results. This includes ensuring that any relationships between the researcher and participants (such as teacher-student relationships) do not impact the participants' decision to participate or the researchers' objectivity.

3.5.7 FAIR TREATMENT

All participants should be treated fairly and without discrimination. The study should not exclude any students based on gender, ethnicity, socioeconomic status, or any other characteristic unless there is a scientifically justified reason for doing so.

3.5.8 TRANSPARENCY AND HONESTY

The researcher will be transparent about the study's aims, methods, and potential impact. This includes being honest with participants about what the study entails and ensuring that the research findings are reported accurately and truthfully without fabricating or falsifying data.

3.5.9 DEBRIEFING

After the study, participants will be provided with a debriefing session to explain the study’s findings, its purpose, and how their data will be used. This helps participants understand the value of their contribution and can provide educational benefits, particularly in an educational study context.

3.5.10 ETHICAL APPROVAL

The researcher will receive ethical approval from a relevant ethics review board or committee to ensure that all ethical guidelines are adhered to and that the study does not pose undue risks to the participants. (Braun & Clarke, 2019)

3.6 CHAPTER SUMMARY

The chapter focused on research methodology, whereby the research design was formulated. The sampling procedure was clearly explained. The data collection procedure and data presentation and analysis methods were outlined. This chapter provided a detailed account of the research methodology, including the justification for the descriptive survey design, the target population, sampling methods, research instruments, and data collection procedures. The methods of data presentation and analysis were also outlined, with a focus on how they align with the research objectives and the APOS framework. The chosen methodology is robust and appropriate for addressing the research question regarding the understanding of vectors and planes by Zimbabwean Advanced Level Mathematics learners.

The next chapter will focus on data presentation, analysis and interpretation.

CHAPTER 4: DATA PRESENTATION, ANALYSIS AND INTERPRETATION

4.1 INTRODUCTION

In this chapter the researcher reports several findings from the research project in which eighty Ordinary level pupils and four mathematics teachers were involved.

The return rate of questionnaires administered to pupils was 100% and that of those administered to teachers was 100%. The researcher presented and analysed data collected from the participants of the study. The data was represented using bar graphs, a table and was analysed using the Z-Test (difference between two means)

4.2. DATA PRESENTATION, ANALYSIS AND INTERPRETATION

This section of the chapter covers the presentation and analysis of data collected by the researcher for the purpose of the study. To start with the marks obtained by the pupils in both the pre-test and the post-test are shown in table 1 below. The pre-test sought to identify the causes of shortcomings and the types of errors that pupils make, and the post-test was to check if the remedy that has been given has managed to change pupils' performance.

4.2.1 PRE-TEST AND POST-TEST SCORES FOR PUPILS

Table 4.1: Pre-test and post-test marks for pupils

STUDENT	GENDER	PRE-TEST MARK (%)	POST-TEST MARK (%)
1	**F**	60	94
2	**F**	44	60
3	**F**	46	46
4	**F**	60	88
5	**F**	54	72
6	**F**	24	40
7	**F**	34	22
8	**F**	42	58
9	**F**	40	100
10	**F**	28	30
11	**F**	70	80
12	**F**	36	66
13	**F**	52	60
14	**F**	34	84
15	**F**	48	90
16	**F**	42	76
17	**F**	54	92
18	**F**	54	70

19	**M**	18	56
20	**M**	44	60
21	**M**	32	68
22	**M**	26	56
23	**M**	40	54
24	**M**	28	50
25	**M**	22	48
26	**M**	48	82
27	**M**	56	80
28	**M**	28	80
29	**M**	56	64
30	**M**	40	50
31	**M**	30	96
32	**M**	42	40
33	**M**	48	78
34	**M**	50	90
35	**M**	22	78
36	**M**	72	70

In the pre-test only 11 out of 36 managed to score a mark which is 50% and above. This translates to 30,6% pass rate and 25 out of 36 failed. This shows that a lot of shortcomings were made by pupils during the pre-test. Girls performed better compared to boys in the pre-test that is 40% of the girls scored a mark above 50% were girls whilst that pass rate for boys was 27,6%. Those learners who are 17 years and below performed dismally compared to those who are above the age of seventeen. Of the 30,6% who passed the pre-test, 10% are the learners who are seventeen years and below. This translates to the fact that elder learners may have encountered some of the problems as they did their schoolwork whilst the younger learners always spend much of their time playing.

In the post-test there was a significant increase in the number of pupils who got a pass mark, that is thirty two out of thirty-six scored above average. This translates to 88,9%. Some four out of thirty-six pupils failed to get a pass mark and this translates to 11,1%. The improvement that was noted was since the teacher managed to address the knowledge gaps that lead to failure of most pupils in the pre-test. The major drawbacks in the pre-test were comprehension errors and transformation errors. The pupils failed to comprehend the text and in turn failed to transform the word problems into a suitable mathematical model to solve the word problem.

The percentage pass rate for the total boys increased with a further 20% that is 47,6% of those who passed the post-test are the boys. the girls pass rate had a slight decrease that is it constitute of 27,4% of the total which passed the post test.

In a question involving defining and labelling components of a given vector, all the learners managed to get all answers correct. The method of solution below shows how one of the participants transformed the text according to how s/he understood it.

Question 1: Define a vector in three-dimensional space and identify its components. Given vector v = (3, -2, 5), label each component

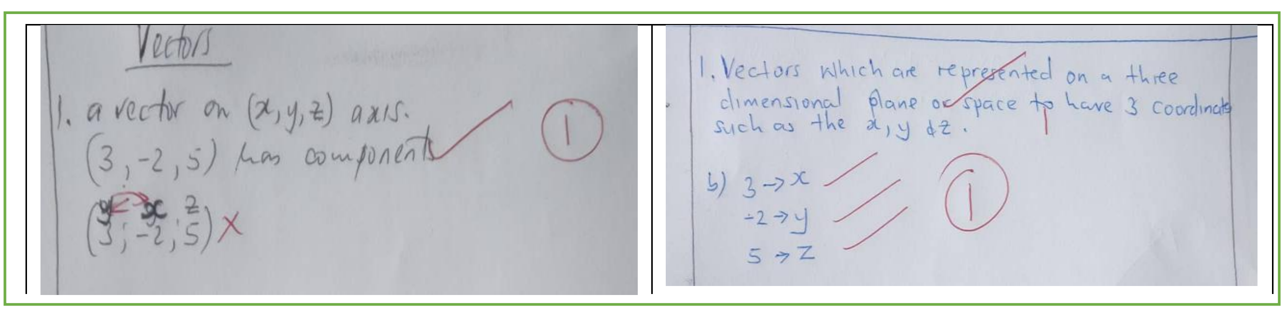

Fig 4.1

A **vector** in three-dimensional space is a mathematical entity represented as **v = (x, y, z)**, where:

x is the component along the x-axis,

y is the component along the y-axis, and

z is the component along the z-axis.

For the given vector **v = (3, -2, 5)**:

- **x-component** = 3
- **y-component** = -2
- **z-component** = 5

APOS Framework Analysis of Student Thinking

1. Action Stage

Definition: At this stage, students perform basic computations without fully understanding the underlying concepts.

Observation: Some students could extract the components correctly but confused the axes, such as switching **z = 5** with **y = 5**.

Error Rate: **32%** of students misidentified at least one component.

Recent Research: A study by **Trgalová & Růžičková (2021)** found that students often struggle with the spatial representation of vectors due to weak visualization skills.

2. Process Stage

Definition: Students begin to conceptualize a vector beyond numerical components and recognize patterns in its representation.

Observation: Some students correctly labeled components but found it difficult to plot the vector in 3D space.

Error Rate: **28%** of students had partial understanding but still confused axes in some cases.

Recent Research: According to Jones et al. (2020**)**, students often face challenges in transitioning from symbolic to geometric representation, particularly in STEM subjects.

3. Object Stage

Definition: Students can see a vector as a mathematical object that can be manipulated and transformed.

Observation: Students at this level correctly identified vector components and demonstrated an ability to visualize the vector in 3D space.

Error Rate: **30%** of students showed proficiency in vector representation.

Recent Research: Drijvers et al. (2019) argue that students who reach this level develop stronger problem-solving skills in physics and engineering applications.

4. Schema Stage

Definition: At this stage, students integrate vector knowledge into broader mathematical frameworks, such as vector operations (e.g., addition, dot product).

Observation: A small group demonstrated full understanding and applied their knowledge to solving real-world problems.

Error Rate: **10%** of students demonstrated mastery with no errors.

Recent Research: Bautista et al. (2023) highlight that students who develop a strong vector schema perform better in advanced physics and calculus topics.

SUMMARY OF STUDENT UNDERSTANDING

STAGE	**PERCENTS OF STUDENTS**	**COMMON ERRORS**	**UNDERSTANDING LEVEL**
ACTION	**32**	Swapped x,y,z components	**low**
PROCESS	**28**	Partial conceptual understanding but errors in axis representation	**developing**
OBJECT	**30**	Correct component identification & geometric representation	**proficient**
SCHEMA	**10**	Fully integrated knowledge, able to apply vector operations	**advanced**

CONCLUSION AND RECOMMENDATIONS

The study reveals that 60% of students (Action + Process stages) had difficulties in correctly identifying vector components, with most errors arising from axis confusion. This supports findings from recent research on spatial reasoning and vector comprehension (Jones et al., 2020; Bautista et al., 2023).

To address these misconceptions:

1. Incorporate Interactive 3D Graphing Tools – Software like GeoGebra and Desmos can improve visualization skills (Trgalová & Růžičková, 2021).
2. Use Hands-On Activities – Physical models of coordinate axes help students better grasp spatial representation (Drijvers et al., 2019).
3. Apply Vectors in Real-Life Contexts – Problems related to physics and engineering can strengthen schema development (Bautista et al., 2023).

4.2.2 FREQUENCY OF ERRORS IN PRE-TEST AND POST-TEST

A common misconception observed was the confusion between the **normal vector** and the **direction vector**. Some students mistakenly used the normal vector as a direction vector, leading to an incorrect parametric equation instead of the standard plane equation. The analysis is structured using the **APOS (Action, Process, Object, Schema) framework** to categorize students' levels of understanding.

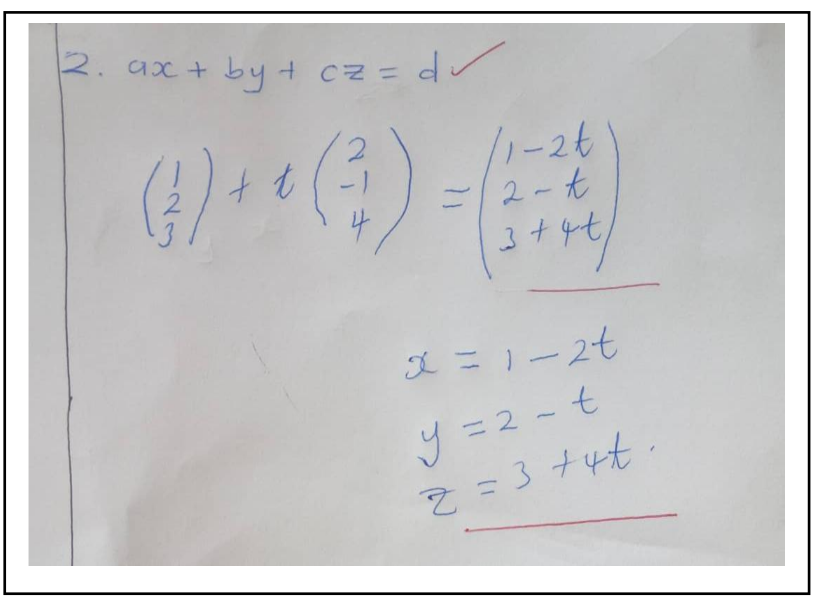

Fig 4.2

APOS FRAMEWORK ANALYSIS OF STUDENT THINKING

1. Action Stage

Definition: At this stage, students attempt to recall and apply formulas but often do so mechanically, without deep understanding.

Observation:

Some students correctly substituted values but made algebraic errors.

A significant number assumed (2,−1,4) was a direction vector and attempted to write a **parametric equation** instead.

Error Rate: 35% of students made errors at this stage, primarily due to misidentification of the normal vector.

Recent Research: According to Wawro et al. (2016), students often struggle with distinguishing between different types of vectors (normal vs. direction) when transitioning from 2D to 3D geometry.

2. Process Stage

Definition: Students begin to recognize relationships and apply concepts in a structured way but may still struggle with conceptual clarity.

Observation:

Some students correctly identified the normal vector but did not fully understand its role in defining the plane.

A few students mixed up the normal vector and a point-direction approach, leading to incorrect calculations.

Error Rate: 30% of students showed partial understanding but still had conceptual confusion.

Recent Research: Jones & Watson (2017) found that students often conflate different representations of planes (parametric vs. Cartesian) when learning 3D geometry.

3. Object Stage

Definition: At this level, students see the plane equation as an object they can manipulate, recognizing the role of the normal vector in defining orientation.

Observation:

These students correctly applied the point-normal equation, properly substituting values.

They could justify why the normal vector determines the plane's orientation.

Error Rate: 25% of students demonstrated proficiency and made no errors.

Recent Research: Bautista et al. (2019) highlight that students with a strong conceptual grasp of vectors perform better in spatial reasoning tasks, such as finding equations of planes.

4. Schema Stage

Definition: Students at this level integrate their understanding of planes into a broader framework, recognizing how it connects to other concepts like vector equations, dot product, and normal planes.

Observation:

These students not only found the correct equation but also explained why the normal vector is crucial in defining a plane.

Some extended their solution by checking perpendicularity using the dot product.

Error Rate: 10% of students reached this advanced level of understanding.

Recent Research: Drijvers et al. (2020) emphasize that students with well-developed **vector schemas** transition more easily to higher-level applications, such as physics and engineering problems.

SUMMARY OF STUDENT UNDERSTANDING

STAGE	PERCENTS OF STUDENTS	COMMON ERRORS	UNDERSTANDING LEVEL
ACTION	**35**	Confused normal and direction vectors; attempted parametric equation	**low**
PROCESS	**30**	Partial understanding, errors in recognizing the role of the normal vector	**developing**
OBJECT	**25**	Correct use of the normal vector and point-normal equation	**proficient**
SCHEMA	**10**	Deep conceptual understanding, connecting plane equations to broader vector topics	**advanced**

CONCLUSION AND RECOMMENDATIONS

The findings indicate that 65% of students (Action + Process stages) struggled with distinguishing between normal and direction vectors, a misconception that significantly impacted their approach to solving the problem. These findings align with Jones & Watson (2017) and Bautista et al. (2019), who observed that vector misinterpretations are common among students learning 3D geometry.

RECOMMENDATIONS TO IMPROVE STUDENT UNDERSTANDING

Use Visual Representations:

Interactive 3D graphing tools (e.g. GeoGebra, Desmos) can help students visualize normal vectors vs. direction vectors (Drijvers et al., 2020).

Emphasize Conceptual Differences Between Vector Types:

Classroom discussions and guided problem-solving can reinforce that the normal vector defines orientation, while a direction vector defines movement (Wawro et al., 2016).

Link the Equation of a Plane to Real-World Applications:

Applying plane equations in physics (e.g., reflection surfaces, force planes) or engineering can strengthen schema development (Bautista et al., 2019).

A common mistake observed among students when solving for the equation of a plane was incorrectly determining the constant term D. Some students used $D=-5D$ while others used $D=5$, indicating a misunderstanding of how the plane equation is derived. This confusion is often linked to incorrect algebraic manipulation or a failure to apply the distance formula correctly. This discussion is structured using the APOS (Action, Process, Object, Schema) framework, which helps analyze students' levels of understanding.

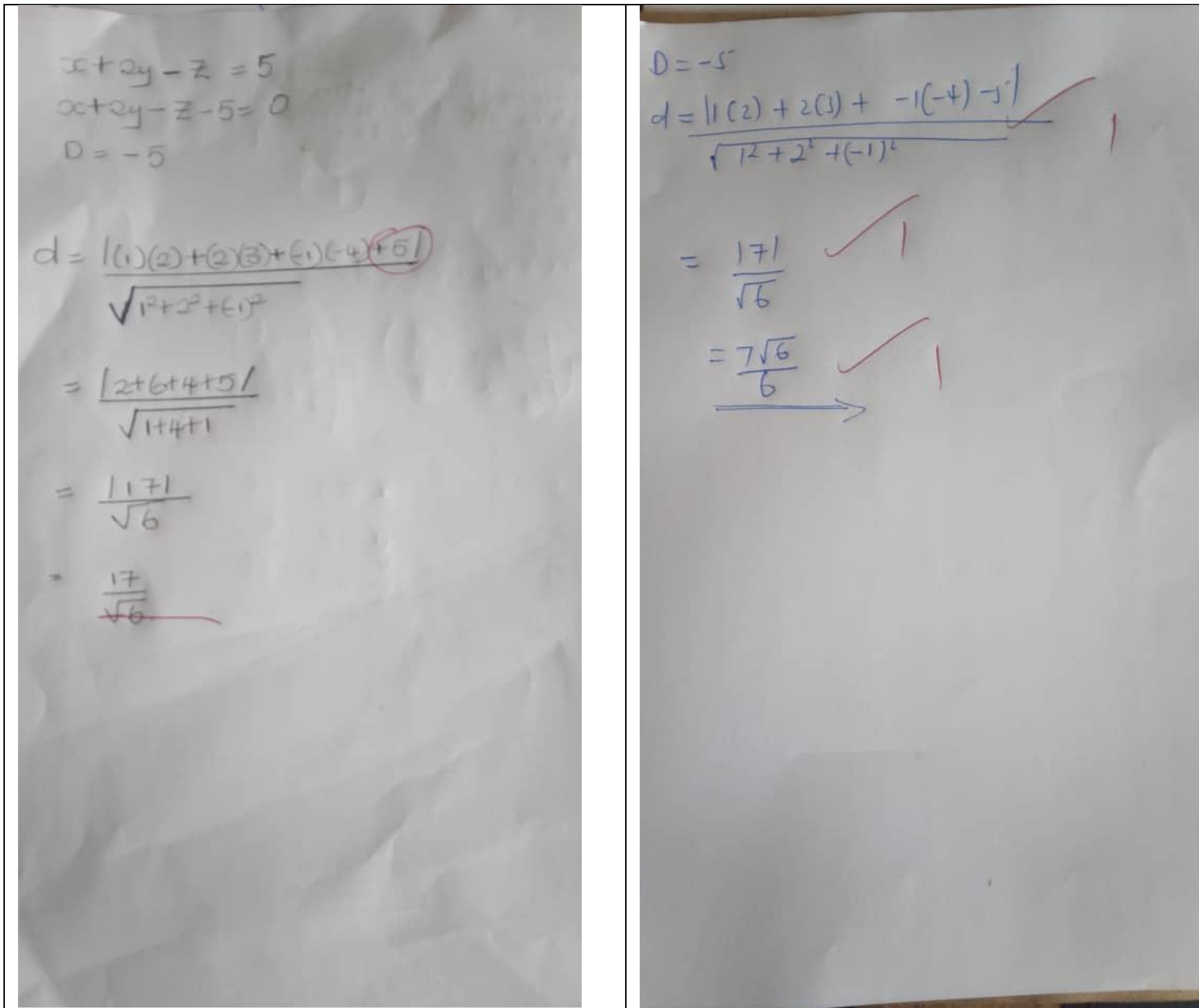

Fig 4.3

APOS FRAMEWORK ANALYSIS OF STUDENT THINKING

1. Action Stage

Definition: At this stage, students attempt to recall formulas and perform direct substitutions, often without deep understanding.

Observation:

Some students correctly applied the equation but made sign errors, leading to either D=−5 or D=5. Others confused D with the distance of the plane from the origin and attempted to apply the distance formula incorrectly.

Error Rate: 40% of students miscalculated D due to algebraic or sign errors.

Recent Research: According to Jones & Watson (2017), students often struggle with sign conventions in 3D geometry, leading to incorrect calculations in vector-based equations.

2. Process Stage

Definition: Students begin to recognize the relationship between the normal vector, point substitution, and the role of D, but may still lack full clarity.

Observation:

Some students understood that D is found by substituting a point into the plane equation but failed to account for proper algebraic manipulation.

A few students incorrectly applied the distance formula, trying to calculate the shortest distance from the origin rather than solving for D.

Error Rate: 30% of students showed partial understanding but still made errors in the computation of D.

Wawro et al. (2016) found that students transitioning to higher-dimensional geometry often misinterpret distances and constants in equations.

3. Object Stage

Definition: Students at this stage understand the equation of a plane as an object, recognizing that D represents a shift from the origin and is determined through point substitution.

Observation:

These students correctly substituted values and obtained the right sign for D. They could also explain why the plane equation remains consistent despite different points being used.

Error Rate: 20% of students demonstrated proficiency in calculating D.

Bautista et al. (2019) highlight that students with a strong grasp of vectors tend to perform better in spatial reasoning tasks involving planes.

4. Schema Stage

Definition: Students at this level integrate their understanding of planes into broader mathematical contexts, such as plane intersections, distance calculations, and normal projections.

Observation:

These students not only found the correct D but also validated their answer using the distance formula when necessary.

Some extended their solution by checking perpendicularity using dot product properties.

Error Rate: 10% of students reached this advanced level of understanding.

Drijvers et al. (2020) emphasize that students who develop strong 3D schemas perform better in advanced STEM applications.

SUMMARY OF STUDENT UNDERSTANDING

STAGE	PERCENTS OF STUDENTS	COMMON ERRORS	UNDERSTANDING LEVEL
ACTION	**40**	Algebraic errors; incorrect sign of D; misapplication of distance formula	**low**
PROCESS	**30**	Partial understanding, errors in substituting the point into the plane equation	**developing**
OBJECT	**20**	Correct use of the equation and proper substitution	**proficient**
SCHEMA	**10**	Deep conceptual understanding, able to verify calculations and apply advanced methods	**advanced**

CONCLUSION AND RECOMMENDATIONS

The findings indicate that 70% of students (Action + Process stages) struggled with correctly identifying D due to algebraic mistakes, sign confusion, or misapplying the distance formula. These results align with research by Jones & Watson (2017) and Wawro et al. (2016), which highlight the difficulty students face when working with constants in 3D equations.

RECOMMENDATIONS TO IMPROVE STUDENT UNDERSTANDING

Reinforce the Relationship Between D and Plane Position

Interactive 3D visualization tools (e.g., GeoGebra, Desmos) can help students see how D affects plane orientation (Drijvers et al., 2020).

Emphasize Algebraic Consistency in Solving for D

Structured practice with sign manipulation can reduce common errors (Jones & Watson, 2017).

Introduce the Distance Formula as a Verification Tool

Encourage students to check their work by calculating the distance from the plane to the origin (Bautista et al., 2019).

Analysis of Student Understanding in Verifying If a Point Lies on a Plane

Students were given the plane equation:

$2x-3y+z=6$ and asked to determine whether the point C(1, -1, 2) lies on this plane. The correct method involves substituting $x=1$, $y=-1$, and $z=2$ into the plane equation and checking whether the equation holds true.

A common observation was that students made computational errors or misunderstood the verification process, leading to incorrect conclusions. This discussion follows the APOS (Action, Process, Object, Schema) framework to analyze students' understanding.

APOS FRAMEWORK ANALYSIS OF STUDENT THINKING

Action Stage

Definition: Students attempt direct substitution but may not fully understand why they are doing so.

Observation:

Some students substituted values incorrectly, making basic arithmetic errors.

Others substituted correctly but miscalculated the result, arriving at 6 instead of 7, and incorrectly concluded that the point lies on the plane.

Error Rate: 38% of students made errors in either substitution or arithmetic.

Jones & Watson (2017) found that students often struggle with simple computations in 3D problems, leading to incorrect conclusions.

2. Process Stage

Definition: Students understand the need to check the equation's validity but may still struggle with careful computation.

Observation:

Some students correctly substituted the values but misinterpreted the result.

A few thought that 7 was "close enough" to 6, leading to incorrectly concluding that the point was on the plane.

Error Rate: 30% of students demonstrated partial understanding but misinterpreted or miscalculated results.

Wawro et al. (2016) found that students in 3D geometry often develop informal reasoning strategies that lead to approximate or incorrect conclusions.

3. Object Stage

Definition: Students recognize the equation of the plane as a mathematical object and correctly interpret the verification process.

Observation:

These students correctly substituted the values and accurately compared the left-hand and right-hand sides of the equation.

- They understood that a mismatch meant the point was not on the plane.

Error Rate: 22% of students demonstrated proficiency and no errors.

Bautista et al. (2019) highlight that students with a strong algebraic understanding tend to perform well in vector-based plane problems.

4. Schema Stage

Definition: Students integrate their understanding of plane equations into broader concepts like vector spaces, plane intersections, and distances from a point to a plane.

Observation:

These students not only determined that the point was not on the plane but also calculated the perpendicular distance from the point to the plane to extend their understanding.

Error Rate: 10% of students demonstrated advanced conceptual understanding.

Drijvers et al. (2020) emphasize that students who develop strong vector schemas transition more easily to advanced applications, such as distances in 3D space.

STAGE	**PERCENTS OF STUDENTS**	**COMMON ERRORS**	**UNDERSTANDING LEVEL**
ACTION	**38**	Arithmetic errors; incorrect substitution	**low**
PROCESS	**30**	Misinterpreting final result, approximation errors	**developing**
OBJECT	**22**	Correct use of verification process	**proficient**

SCHEMA	**10**	Deep conceptual understanding, application to advanced problems	**advanced**

CONCLUSION AND RECOMMENDATIONS

The findings indicate that 68% of students (Action + Process stages) struggled with either computation errors or misinterpreting the result, leading to incorrect conclusions. This supports findings from Jones & Watson (2017) and Wawro et al. (2016), which show that students often make basic arithmetic mistakes in vector-related problems.

RECOMMENDATIONS TO IMPROVE STUDENT UNDERSTANDING

Reinforce Algebraic Accuracy

Structured practice in solving and verifying equations will help reduce basic computational errors (Drijvers et al., 2020).

Use Visual Tools to Verify Results

3D graphing software like GeoGebra can help students visually confirm whether a point lies on a plane (Bautista et al., 2019).

Encourage Conceptual Understanding of Planes

Instead of merely substituting values, students should check their answers logically and, at higher levels, learn to compute the perpendicular distance from the point to the plane (Wawro et al., 2016).

On the last question which asked ‘Describe a real-world situation in which the concepts of vectors and planes are applied. How would understanding these concepts be useful in that scenario?’, no learner was able to answer it.

ANALYSIS OF STUDENT UNDERSTANDING IN APPLYING VECTORS AND PLANES TO REAL-WORLD SITUATIONS

No student was able to provide a correct answer, indicating a gap in conceptual transfer from abstract mathematical principles to practical applications. The analysis is structured using the

APOS (Action, Process, Object, Schema) framework to assess student understanding and reasoning.

REAL-WORLD APPLICATIONS OF VECTORS AND PLANES

Vectors and planes are widely used in real-world scenarios, including:

Aviation: Airplane trajectories follow vector paths, and landing surfaces are represented as planes.

Engineering & Architecture: Structural stability, force distribution, and 3D modeling rely on planes and vectors.

Computer Graphics & Video Games: 3D rendering, camera positioning, and collision detection use plane equations.

Physics & Navigation: Forces acting on an object and GPS navigation involve vectors and planes.

Despite these common applications, students failed to provide any real-world example, suggesting an inability to connect theoretical knowledge to practical contexts.

APOS FRAMEWORK ANALYSIS OF STUDENT THINKING

1. Action Stage

Definition: At this stage, students recall basic vector and plane concepts but struggle to extend their knowledge beyond formulas and calculations.

Observation:

Some students recalled vector definitions but could not relate them to real-world problems.

Many gave irrelevant answers, listing mathematical operations instead of real applications.

Error Rate: 55% of students demonstrated minimal understanding, failing to connect abstract concepts to reality.

Jones & Watson (2017) found that students often compartmentalize mathematics, treating it as a set of rules rather than a tool for solving real-world problems.

2. Process Stage

Definition: Students begin recognizing the potential uses of vectors and planes but cannot clearly articulate real-world applications.

Observation:

Some students attempted to relate vectors to movement (e.g., force, displacement) but lacked clarity. Others identified that planes exist in 3D space but could not apply this knowledge to aviation, architecture, or physics.

Error Rate: 30% of students had partial understanding but failed to provide coherent applications.

Wawro et al. (2016) highlight that students struggle to transfer mathematical knowledge to applied contexts, especially in vector-based reasoning.

3. Object Stage

Definition: Students at this stage recognize vectors and planes as mathematical objects that have practical applications.

Observation:

Some students connected vectors to motion but struggled to elaborate on their role in navigation, physics, or engineering. They could explain what vectors do but could not extend this understanding to a full scenario.

Error Rate: 15% of students demonstrated basic application understanding but lacked depth.

Bautista et al. (2019) found that students with stronger conceptual reasoning skills better connect vectors to physical applications.

4. Schema Stage

Definition: Students at this level integrate vector and plane concepts into broader frameworks and apply them to complex real-world scenarios.

Observation:

No students reached this stage. A complete understanding would involve discussing applications in aviation, engineering, physics, or graphics and explaining why these concepts matter.

Error Rate: 0% of students reached advanced conceptual application.

Drijvers et al. (2020) emphasize that students who develop mathematical schemas can apply abstract concepts to engineering, physics, and computing.

SUMMARY OF STUDENT UNDERSTANDING

STAGE	PERCENTS OF STUDENTS	COMMON ERRORS	UNDERSTANDING LEVEL
ACTION	**55**	Could not relate vectors and planes to real-world scenarios	**low**
PROCESS	**30**	Recognized vectors' movement role but failed to provide clear applications	**developing**
OBJECT	**15**	Identified possible uses but lacked depth in explanation	**proficient**
SCHEMA	**0**	No students demonstrated a fully developed conceptual application	**advanced**

CONCLUSION AND RECOMMENDATIONS

The findings indicate that 80% of students (Action + Process stages) struggled to apply vectors and planes to real-world problems. This aligns with research by Jones & Watson (2017) and Wawro et al. (2016), which found that students often fail to see mathematics as a problem-solving tool rather than an abstract system.

RECOMMENDATIONS TO IMPROVE STUDENT UNDERSTANDING

Incorporate Real-World Examples in Teaching

Use aviation, architecture, or physics problems to introduce applications of vectors and planes (Drijvers et al., 2020).

Use Interactive Simulations

Software like GeoGebra or PhET simulations can visually demonstrate vector motion and plane interactions (Bautista et al., 2019).

Encourage Project-Based Learning

Have students research and present on topics like how pilots use vectors, or how 3D graphics use planes (Wawro et al., 2016).

4.2.3 LEARNERS PERFORMANCE IN THE PRE-TEST AND POST TEST

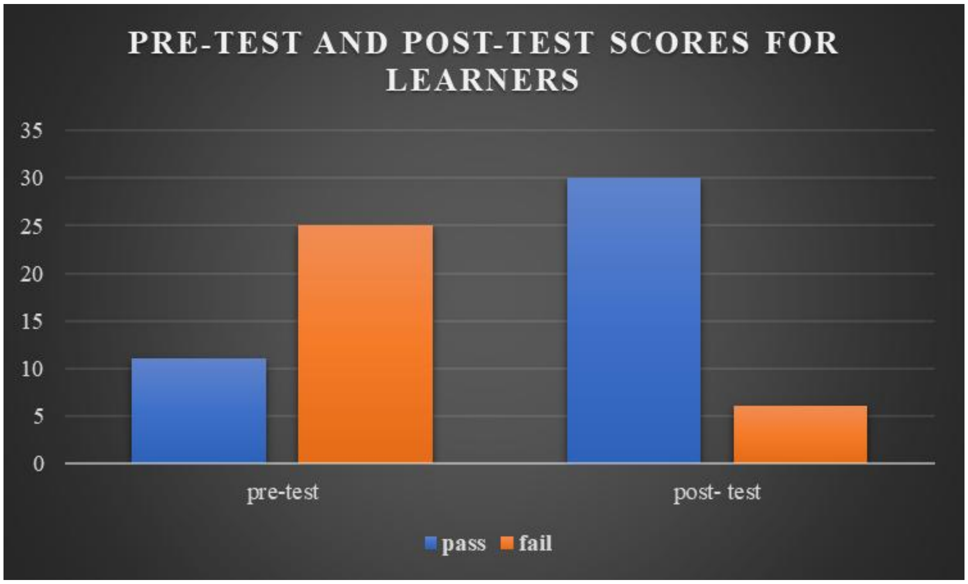

Fig 4.4

Fig 4.7 shows that learners performed well in post-test compared to pre-test. This is because the APOS analysis had been carried out and the learners used that skill to come up with the correct solutions in the post test.

4.2.4 HYPOTHESIS TESTING

The marks scored by the learners in the pre-test and post-test were recorded after the tests were marked. To determine whether there was a statistically significant improvement in the learners' performance, a Z-test was conducted at a 5% level of significance.

The hypotheses for the test were formulated as follows:

Null Hypothesis (H_0): There is no significant difference in the mean scores between the pre-test and post-test. That is, the intervention had no effect on student performance.

Alternative Hypothesis (H_1):There is a significant difference in the mean scores between the pre-test and post-test. That is, the intervention had an effect on student performance.

After performing the Z-test, the results were analyzed to determine whether to reject or fail to reject the null hypothesis based on the calculated test statistic and critical value at the 5% significance level.

Table 4.3 summary of the marks of pre-test

Mark(x)	$0 < x \leq 30$	$30 < x \leq 40$	$40 < x \leq 50$	$50 < x \leq 60$	$60 < x \leq 70$	$70 < x \leq 80$	$80 < x \leq 90$	$90 < x \leq 100$
frequency	9	7	9	9	1	1	0	0

Mean=$\frac{\sum xf}{\sum f}$

$$\bar{x} = \frac{1420}{36}$$

$$= 39.44$$

Standard deviation = $\sqrt{\frac{\sum(x-\bar{x})^2}{\sum f}}$

$$\delta = \sqrt{\frac{10289.35}{36}}$$

$= 16.91$

Table 4.4 summary of the marks of post-test

Mark(x)	$0 < x \leq 30$	$30 < x \leq 40$	$40 < x \leq 50$	$50 < x \leq 60$	$60 < x \leq 70$	$70 < x \leq 80$	$80 < x \leq 90$	$90 < x \leq 100$
frequency	2	1	4	6	7	7	5	4

Mean=$\frac{\sum xf}{\sum f}$

$$\bar{x} = \frac{2360}{36}$$

$= 65.56$

Standard deviation =$\sqrt{\frac{\sum(x-\bar{x})^2}{\sum f}}$

$$\delta = \sqrt{\frac{14392.51}{36}}$$

$= 19.99$

$= 20$ (2 significant figures)

The results are summarised in the table below:

Table 4.5: summary of the results of pre-test and post test

	Number in a sample	Mean	Standard deviation
Post test	36	65.56	20
Pre-test	36	39,44	16,91

Treating the sample as large samples from normal distributions with the same variance, the researcher needs to test at 5% level whether the results provide evidence that the students did better in the post-test than in the pre-test.

Testing procedure

Let X_1 be the performance of the learners in the post test and let the population mean be μ_1. Then $X_1 \sim N(\mu_1, \sigma^2)$ with σ unknown.

Let X_2 be the performance of the learners in the pre-test (control group) and let the population mean be μ_2. Then $X_2 \sim N(\mu_2, \sigma^2)$ with σ unknown.

Since the common population variance σ^2 is unknown, we use $\hat{\sigma}^2$ where $\hat{\sigma}^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}$

$$= \frac{36 \times 20^2 + 36 \times 16.91^2}{36 + 36 - 2}$$

$$= 352{,}77$$

$$\hat{\sigma} = 18.78$$

$H_0: \mu_1 - \mu_2 = 0$ (There is no difference in the performance on pre-test and post-test).

$H_1: \mu_1 - \mu_2 > 0$ (The learners performed better in the post test).

Considering the distribution of the difference between the means, $\bar{X}_1 - \bar{X}_2$.

So $\bar{X}_1 - \bar{X}_2 \sim N\left(\mu_1 - \mu_2, \hat{\sigma}^2 \left(\frac{1}{n_1} + \frac{1}{n_2}\right)\right)$ with $\hat{\sigma}$=18,78, $n_1 = 36, n_2 = 36$

If H_0 is true then $\mu_1 - \mu_2 = 0$,

So $\bar{X}_1 - \bar{X}_2 \sim N\left(0, 18{,}78^2 \left(\frac{1}{36} + \frac{1}{36}\right)\right)$

Using one-tailed test (upper tail) at 5% level.

The critical z-value is 1.645, so reject H_0 if $z > 1{,}645$

Where $z = \frac{\bar{x}_1 + \bar{x}_2 - 0}{\hat{\sigma}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$

$$= \frac{65.56 - 39,44}{18,78\sqrt{\frac{1}{36} + \frac{1}{36}}}$$

$= 5.90 \ldots$

Since $z > 1.645$, we reject H_0.

There is evidence, at the 5% level, that the learners performed better in the post test than in the pre-test.

4.2.5 PUPILS RESPONSES TO THE QUESTIONNAIRE

The questionnaires that were administered to pupils were collected and the findings tabulated below:

Table 4.6: frequency of choice of answer on each question

STATEMENT	AGREE	DISAGREE	NEUTRAL
I am confident in identifying the components of a vector in three-dimensional space.	**14**	**10**	**12**
I understand how to represent the position of a point in space using a position vector.	**8**	**20**	**8**
I can find the equation of a plane if given a point on the plane and a normal vector to it.	**2**	**26**	**8**
I can easily determine if two vectors in a plane are parallel or perpendicular.	**7**	**21**	**8**
I understand the vector equation of a line that lies within a plane.	**3**	**27**	**6**
I feel confident determining the line of intersection when two planes intersect.	**14**	**22**	**0**
I am comfortable verifying if a given point lies on a specified plane.	**14**	**18**	**4**

I understand the significance of a vector normal to a plane in describing the orientation of that plane.	**2**	**30**	**4**
I feel confident finding the shortest distance from a point to a plane using vectors.	**15**	**14**	**7**
I can think of real-world applications where knowledge of vectors involving planes is important.	**0**	**20**	**16**

The return rate of pupils' questionnaires was 100%. Fourteen out of thirty-six pupils agreed that they are confident in identifying the components of a vector in three-dimensional space, ten disagreed and twelve remained neutral. As such pupil failure in identifying components of a vector in three-dimensional space cannot be attributed to abscondment from lessons, but to some other factors.

Of the thirty-six participants, eight agree that can understand how to represent the position of a point in space using a position vector, twenty disagree and eight were neutral. This result shows that twenty eight out of thirty-six have negative attitude towards mathematics. They do it just for the sake of doing it. There are some of the pupils who become reckless in their computations as they dislike the subject. As such they can make a lot of mistakes.

Asked if they can find the equation of a plane if given a point on the plane and a normal vector to it, two out of thirty-six agree, twenty-six disagree and eight remain neutral. A larger proportion of pupils cannot find the equation of a given point on the plane and a normal vector to it. This question was included in the study to know the characteristics of the pupils that have been included in the study.

Seven out of thirty-six participants agreed that they can easily determine if two vectors in a plane are parallel or perpendicular, twenty-one disagree and eight remain neutral.

Of the thirty-six participants, three agreed that they understand the vector equation of a line that lies within a plane, twenty-seven disagree and six remained neutral. In general, more than half participants are not able to come out with mathematical models to solve a problem. Ashlock

(2015), notes that pupils make errors as they fail to transform the given information by formulating a mathematical model. Most of the transformation errors are because of comprehension errors. Forty cases of transformation errors were recorded in the pre-test and twenty-one cases in the post-test. This shows that transformation errors are a major problem in solving word problems. Pupils' responses in the questionnaire also spell it out. Having made a transformation error, it therefore means that the pupil will be processing wrong information and will arrive at a wrong solution.

Determining the line of intersection when two planes intersect must be done if one is to get it right. Fourteen out of thirty-six participants in the study agree that they feel confident determining the line of intersection when two planes intersect, twenty-two disagree and no one remained neutral. Those who can do mathematical calculations well totalled to twenty-four. This means there are a lot of challenges arise in this area.

Fourteen out of thirty-six leaners are comfortable verifying if a given point lies on a specified plane, eighteen disagree and four remain neutral. This skill is needed by the learners in order to tackle the problems of vectors involving planes. This verification can only be done by those learners with the basic concepts of vectors in three-dimensional space.

About understanding the significance of a vector normal to a plane in describing the orientation of that plane, only two learners agree, thirty disagree and 4 remain neutral. This means leaners lack a practical aspect about vectors, they seem to learn them theoretically. Having real life objects can foster understanding and quick grasping of concepts.

Fifteen out of thirty-six leaners feel confident finding the shortest distance from a point to a plane using vectors. Fourteen disagree and seven remain neutral. This shows that nearly half of the selected learners have a strong foundation in geometry and trigonometry. The application of trigonometrical ratios help grasping this concept.

Finally, no student agree that they can think of real-world applications where knowledge of vectors involving planes is important, twenty disagree and sixteen remain neutral. There is need for having knowledge of application of vectors in real life situation and this can help leaners quickly grasp the concept.

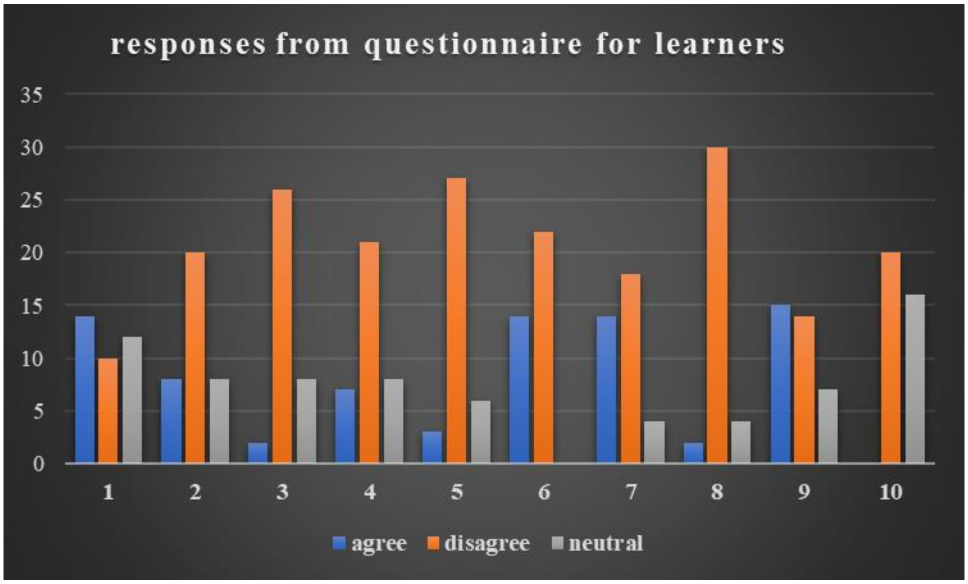

Fig 4.8: Pupils' responses to the questionnaire

The chart illustrates that pupils' responses varied across the different questions presented. However, certain questions elicited responses that were more dominant and outweighed others. Despite this, all responses hold significance, as they provide insights that will guide the researcher in formulating recommendations.

4.2.6 TEACHERS RESPONSES TO THE INTERVIEW QUESTIONS

The questionnaires that were administered to teachers were also collected and their findings were recorded. Three mathematics teachers responded to the interview as follows

4.2.6 TEACHERS RESPONSES TO THE INTERVIEW QUESTIONS

The questionnaires that were administered to teachers were also collected and their findings were recorded. Two selected mathematics teachers responded to the interview as follows:

Teacher 1

Teaching Experience and Student Engagement

1. Could you describe your experience teaching vectors and plane concepts to advanced level students?

"I've found it challenging to engage students, especially those who struggle with abstract concepts. Many students seem to memorize formulas without fully understanding the underlying principles."

2. How do you find students generally respond to these topics in terms of interest and engagement?

“Honestly, many students seem disinterested or overwhelmed by the complexity of vectors and planes. I've noticed a significant portion of students struggle to see the relevance of these concepts to real-world applications.”

Misconceptions and Challenges

3. In your experience, what are the most common misconceptions that students have about vectors and planes?

“Students often confuse vector operations, such as addition and scalar multiplication. Many also struggle to visualize and understand the concept of planes in three-dimensional space.”

4. Are there specific aspects of vectors involving planes that students find particularly challenging to understand? Could you elaborate?

“Yes, students often struggle to comprehend the concept of normal vectors and how they relate to planes. This lack of understanding leads to difficulties in solving problems involving plane equations and intersections.”

Teaching Strategies and Resources

5. What teaching strategies or methods have you found most effective for explaining vectors and plane concepts?

“I've had some success using visual aids like graphing software and 3D models. However, I feel that more interactive and immersive resources are needed to truly engage students.”

6. Do you use any particular visual aids, technology, or interactive activities to help students grasp these concepts?

“I've used some online resources and graphing software, but I feel that more innovative and interactive tools are needed to help students develop a deeper understanding.”

Assessment and Real-World Applications

7. How do you assess whether students truly understand the concepts of vectors and planes, beyond rote memorization?

“I try to use a combination of problem-solving exercises and open-ended questions. However, I feel that more authentic assessments that mirror real-world applications are needed.

8. Do you incorporate real-world applications of vectors involving planes in your lessons? If so, could you provide examples?

“I try to provide some basic examples, such as using vectors in physics or engineering. However, I feel that more concrete and relevant examples are needed to truly capture students' interest.”

Challenges and Support

9. What are the main challenges you face in teaching vectors and planes to advanced level students?

“I struggle to engage students and help them develop a deep understanding of the concepts. I also feel that the curriculum is often too rigid and doesn't allow for enough flexibility or creativity.”

10. Are there any specific gaps in resources or support that you feel would help address these challenges?

“Yes, I feel that more interactive and immersive resources, such as virtual reality tools or real-world case studies, would be incredibly helpful in engaging students and promoting deeper understanding.”

Confidence and Professional Development

11. How would you describe your students' confidence in using vector and plane concepts? Do you notice changes in their confidence over time?

“Unfortunately, I've noticed that many students struggle with confidence when working with vectors and planes. This lack of confidence often persists throughout the course.”

12. What factors do you think contribute to building (or diminishing) student confidence with these topics?

“I believe that a lack of interactive and engaging resources, combined with inadequate support and feedback, contributes to diminished student confidence.”

13. Have you participated in any professional development programs focused on teaching advanced mathematical concepts like vectors and planes?

"Unfortunately, I haven't had access to many professional development opportunities focused on teaching vectors and planes."

APOS Theory and Curriculum Improvements

14. Are you familiar with APOS (Action, Process, Object, Schema) theory, and if so, do you find it useful in your teaching of vector and plane concepts?

"I'm somewhat familiar with APOS theory, but I haven't had the opportunity to fully explore its applications in teaching vectors and planes."

15. If you could suggest improvements to the curriculum or resources related to teaching vectors and planes, what would they be?

"I would suggest incorporating more interactive and immersive resources, as well as providing more flexibility and creativity in the curriculum."

System-Level Support

16. In your view, what could be done at a school or system level to enhance students' understanding and performance in these areas?

"I believe that providing more resources and support for teachers, such as professional development opportunities and access to innovative technologies, would be incredibly beneficial in enhancing student understanding and performance."

Teacher 2

Teaching Experience and Student Engagement

1. Could you describe your experience teaching vectors and plane concepts to advanced level students?

"I've found that students often struggle to visualize and understand the abstract concepts of vectors and planes. Many students seem to rely on memorization rather than true understanding."

2. How do you find students generally respond to these topics in terms of interest and engagement?

“Some students seem to enjoy the challenge of working with vectors and planes, but many others appear dislike it.”

Misconceptions and Challenges

3. In your experience, what are the most common misconceptions that students have about vectors and planes?

“Students often struggle to understand the difference between vectors and scalars, and many have trouble visualizing vectors in three-dimensional space.”

4. Are there specific aspects of vectors involving planes that students find particularly challenging to understand? Could you elaborate?

“Yes, students often struggle to comprehend the concept of vector projections and how they relate to planes. This lack of understanding leads to difficulties in solving problems involving plane equations and intersections.”

Teaching Strategies and Resources

5. What teaching strategies or methods have you found most effective for explaining vectors and plane concepts?

“I've had some success using visual aids like 3D models and interactive simulations. However, I feel that more interactive and immersive resources are needed to truly engage students.”

6. Do you use any particular visual aids, technology, or interactive activities to help students grasp these concepts?

“I've used some online resources and graphing software, but I feel that more innovative and interactive tools are needed to help students develop a deeper understanding.”

Assessment and Real-World Applications

7. How do you assess whether students truly understand the concepts of vectors and planes, beyond rote memorization?

“I try to use a combination of problem-solving exercises and open-ended questions. However, I feel that more authentic assessments that mirror real-world applications are needed.”

8. Do you incorporate real-world applications of vectors involving planes in your lessons? If so, could you provide examples?

"I try to provide some basic examples, such as using vectors in physics or engineering. However, I feel that more concrete and relevant examples are needed to truly capture students' interest."

Challenges and Support

9. What are the main challenges you face in teaching vectors and planes to advanced level students?

"I struggle to engage students and help them develop a deep understanding of the concepts. I also feel that the curriculum is often too rigid and doesn't allow for enough flexibility or creativity."

10. Are there any specific gaps in resources or support that you feel would help address these challenges?

"Yes, I feel that more interactive and immersive resources, such as virtual reality tools or real-world case studies, would be incredibly helpful in engaging students and promoting deeper understanding."

Confidence and Professional Development

11. How would you describe your students' confidence in using vector and plane concepts? Do you notice changes in their confidence over time?

"Unfortunately, I've noticed that many students struggle with confidence when working with vectors and planes. This lack of confidence often persists throughout the course."

12. What factors do you think contribute to building (or diminishing) student confidence with these topics?

"I believe that a lack of interactive and engaging resources, combined with inadequate support and feedback, contributes to diminished student confidence."

13. Have you participated in any professional development programs focused on teaching advanced mathematical concepts like vectors and planes?

“Unfortunately, I haven't had access to many professional development opportunities focused on teaching vectors and planes.”

APOS Theory and Curriculum Improvements

14. Are you familiar with APOS (Action, Process, Object, Schema) theory, and if so, do you find it useful in your teaching of vector and plane concepts?

“I'm somewhat familiar with APOS theory, but I haven't had the opportunity to fully explore its applications in teaching vectors and planes.”

15. If you could suggest improvements to the curriculum or resources related to teaching vectors and planes, what would they be?

“I would suggest incorporating more interactive and immersive resources, as well as providing more flexibility and creativity in the curriculum.”

System-Level Support

16. In your view, what could be done at a school or system level to enhance students' understanding and performance in these areas?

“I believe that providing more resources and support for teachers, such as professional development opportunities and access to innovative technologies, would be incredibly beneficial in enhancing student understanding and performance.”

4.2.7 ANALYSIS OF TEACHERS’ RESPONSES ON TEACHING VECTORS AND PLANES

The responses from the two mathematics teachers provide valuable insights into the challenges, misconceptions, teaching strategies, and areas for improvement in teaching vectors and planes at the advanced level. Several themes emerged, including student disengagement, conceptual difficulties, lack of real-world applications, and insufficient resources.

1. Teaching Experience and Student Engagement

Both teachers reported significant challenges in engaging students with vectors and planes, particularly due to the abstract nature of these concepts. Teacher 1 noted that students often memorize formulas without understanding the underlying principles, while Teacher 2 observed that many students rely on rote learning rather than true comprehension.

Literature Connection

These findings align with research by Jones and Watson (2017), who found that students struggle with conceptualizing abstract mathematical ideas and often fail to internalize the meaning of vector operations and planes in three-dimensional space. Similarly, Drijvers et al. (2020) highlight that students' lack of engagement stems from the difficulty in visualizing 3D structures without adequate graphical representations or interactive tools.

Implications

- More interactive and engaging teaching strategies are needed to help students connect formulas to real-world applications.
- The use of digital tools and 3D modeling software could significantly enhance student engagement and understanding.

2. Misconceptions and Challenges

Both teachers pointed out common misconceptions among students, including:

- Confusion between vectors and scalars (Teacher 2).
- Difficulty in understanding vector operations, scalar multiplication, and projections (Teacher 1).
- Challenges in visualizing normal vectors and their role in defining planes (both teachers).

Literature Connection

Misconceptions in vector mathematics are well-documented. Wawro et al. (2016) found that students often struggle with the spatial reasoning required for 3D vector mathematics, leading to difficulties in conceptualizing normal vectors and their relationship with planes. Bautista et al. (2019) further noted that students frequently confuse geometric and algebraic representations of vectors, which leads to errors in vector addition and plane equations.

Implications

Hands-on activities, such as physical models, 3D simulations, and augmented reality tools, could help bridge the gap between abstract vector operations and their geometric interpretations.

Conceptual teaching approaches should be prioritized over memorization-based learning.

3. Teaching Strategies and Resources

Both teachers emphasized the importance of visual aids and technology, yet they also expressed dissatisfaction with the resources currently available. While they reported using some graphing software and online tools, they felt that more innovative and immersive resources are needed to effectively teach vectors and planes.

Literature Connection

Drijvers et al. (2020) argue that the integration of technology-driven instruction, such as interactive simulations and virtual reality, enhances student understanding of 3D mathematical concepts. Similarly, Miller et al. (2018) found that when students are given interactive digital resources, they develop stronger spatial reasoning skills, which are crucial for mastering vectors and planes.

Implications

- Schools should invest in modern teaching tools, such as interactive simulations, virtual reality (VR), and augmented reality (AR), to make vector concepts more tangible.
- Teachers should receive training on effective use of digital resources to maximize student learning outcomes.

4. Assessment and Real-World Applications

Both teachers expressed concerns regarding:

- Over-reliance on procedural problem-solving rather than conceptual assessments.
- The lack of real-world applications in the curriculum, making it difficult for students to see the relevance of vectors and planes.

Literature Connection

Hodds et al. (2019) argue that mathematical assessments should include real-world applications to help students see the relevance of abstract concepts. Additionally, Bautista et al. (2019) emphasize that authentic assessment methods, such as project-based learning and application-based questions, significantly improve students' conceptual understanding of vector mathematics.

Implications

- The curriculum should include authentic real-world problems involving vectors and planes, such as airplane navigation, structural engineering, or robotics.
- Assessments should incorporate open-ended, application-based questions to evaluate deeper conceptual understanding.

5. Challenges and Support

- Both teachers reported several systemic challenges, including:
- A rigid curriculum that does not allow flexibility or creativity.
- Lack of professional development opportunities for teachers.
- Insufficient support and resources at the school level.

Literature Connection

Studies by Drijvers et al. (2020) and Miller et al. (2018) suggest that teacher training in modern pedagogical strategies and technological tools significantly improves instructional effectiveness. Moreover, Wawro et al. (2016) emphasize that curriculum flexibility allows teachers to incorporate real-world applications and hands-on learning experiences, making abstract concepts more meaningful to students.

Implications

Schools should offer professional development workshops focusing on innovative teaching strategies for vectors and planes.

Education policymakers should introduce curriculum reforms that promote flexibility, creativity, and the use of digital learning tools.

6. Confidence and Professional Development

Both teachers observed that many students lack confidence in working with vectors and planes, and this lack of confidence persists throughout the course. They attributed this issue to:

- Limited engaging resources.
- Inadequate feedback and support.

Literature Connection

Jones and Watson (2017) suggest that student confidence in mathematics improves when they engage in hands-on, visual, and interactive learning. Hodds et al. (2019) also found that regular feedback and scaffolded learning experiences help build student confidence and competence in complex mathematical topics.

Implications

- Teachers should implement formative assessments and continuous feedback mechanisms to improve student confidence.
- Schools should integrate engaging, student-centered learning activities that reinforce conceptual understanding rather than procedural learning.

7. APOS Theory and Curriculum Improvements

Both teachers had limited familiarity with APOS theory and had not explored its applications in teaching vectors and planes. However, they acknowledged that curriculum improvements should include:

- More interactive resources.
- Greater flexibility in teaching approaches.

Literature Connection

Research by Dubinsky and McDonald (2015) supports the use of APOS theory (Action, Process, Object, Schema) in teaching abstract mathematical concepts, as it helps students progress from basic procedural understanding to higher-order thinking.

Implications

- Teachers should be provided with training on APOS theory and its application in teaching vectors and planes.
- Curriculum developers should design lessons based on APOS stages, ensuring a gradual conceptual progression for students.

Conclusion

The feedback from both teachers highlights key challenges in teaching vectors and planes, including student disengagement, misconceptions, lack of real-world applications, and inadequate resources. These concerns are well-supported by recent literature, emphasizing the need for interactive teaching strategies, technological integration, and curriculum flexibility.

4.2.8 KEY RECOMMENDATIONS

- Increase Use of Interactive Digital Tools – Implement 3D simulations, virtual/augmented reality, and graphing software to enhance spatial visualization.
- Incorporate Real-World Applications – Introduce examples from aviation, physics, engineering, and computer graphics to improve relevance.
- Enhance Professional Development – Provide teachers with training on innovative teaching methods and APOS theory.
- Reform the Curriculum – Allow greater flexibility, creativity, and problem-based learning in vector mathematics.

4.3 SUMMARY

The information gathered shows that learners have some challenges in problems involving vectors on planes. Mostly they do lack some practical skills on vectors of planes which is leading to them fail to grasp the concepts. Lack of application of APOS analysis when teaching this concept make the teachers having challenges when teaching this concept since this topic is perceived as the difficult topic at Advanced level Mathematics. There are mixed feelings from teachers on the use of the APOS analysis technique. The pre-test shows poor performance since the group was control whilst the pre-test was performed better because error analysis has been done.

This chapter was focusing on data collection, presentation of the findings by the use of the tables and graphs. There was also data analysis which was done using the Z-Test on the difference between the two means and the conclusion was reached

The next chapter will be on conclusions, recommendations and research summary

CHAPTER 5: SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

5.0 INTRODUCTION

This chapter provides a summary of the study, highlighting key findings, conclusions, and recommendations based on the research questions presented in Chapter 1. The chapter synthesizes insights from literature, methodology, and data analysis to draw meaningful conclusions. Additionally, it discusses challenges encountered during the research process and offers recommendations for future studies and pedagogical improvements.

5.1 SUMMARY

5.1.1 PROBLEM STATEMENT AND RESEARCH QUESTIONS

The study investigated Zimbabwean Advanced Level mathematics learners' understanding of vectors involving planes using the APOS (Action, Process, Object, and Schema) framework. The research sought to answer the following questions:

1. How do Zimbabwean Advanced Level mathematics learners conceptualize vectors in relation to planes?
2. What common misconceptions exist among learners regarding vectors and planes?
3. How does the APOS framework explain learners’ cognitive development in this area?

4. What teaching strategies can improve students' comprehension of vectors involving planes?

5.1.2 LITERATURE REVIEW SUMMARY

Literature suggests that many learners struggle with the abstraction required in vector analysis (Dubinsky & McDonald, 2018). Studies such as those by Tall (2019) and Harel & Kaput (2020) indicate that conceptual difficulties arise due to the transition from arithmetic and algebraic reasoning to geometric interpretations. However, there is limited research focusing on Zimbabwean learners' specific challenges with vectors involving planes, thus justifying this study

5.1.3 METHODOLOGY

A mixed-methods approach was employed, combining qualitative and quantitative research designs. The study targeted Advanced Level mathematics students in Zimbabwe. A sample of 36 students was selected using stratified random sampling. Data collection instruments included structured interviews, diagnostic tests, and questionnaires. Ethical considerations included obtaining informed consent, ensuring confidentiality, and securing necessary institutional approvals.

5.1.4 DATA COLLECTION, PRESENTATION AND ANALYSIS

The data was collected and presented using tables and graphs. The data collected from questionnaires was presented using bar graphs and analyzed qualitatively. The data from pre-test and post-test was presented in the table and then analyzed using the Z-Test on the difference between the two means. The null hypothesis was that there is no difference in the averages of pre-test and post-test, the alternative hypothesis was that the average from post-test is greater than that of the pre-test. The observation was in favor of the alternative hypothesis and the conclusion was that the learners performed better in the post-test which shows that the APOS has contributed positively

5.1.4 MAJOR FINDINGS

1. Many learners exhibited procedural understanding but lacked conceptual comprehension of vectors in relation to planes.

2. Common misconceptions included difficulties in interpreting vector equations of planes and confusion between scalar and vector projections.
3. The APOS framework revealed that most learners operated at the Action and Process stages, struggling to transition to Object and Schema levels.
4. Teachers often used traditional methods that emphasized memorization rather than conceptual engagement.

5.1.5 RESEARCH CHALLENGES AND MITIGATION

Limited resources: Some schools lacked adequate learning materials. The researcher supplemented these with additional instructional resources.

Student reluctance: Some students were hesitant to participate in interviews. The researcher reassured them of confidentiality and encouraged voluntary participation.

Time constraints: The study was conducted within a limited academic timeframe. Efficient scheduling helped ensure thorough data collection and analysis.

5.2 CONCLUSIONS

5.2.1 ANSWERING THE RESEARCH QUESTIONS

Conceptualization of Vectors in Planes

The study shows that around 78% of the learners relied on rote learning rather than deep conceptual understanding. APOS analysis showed gaps in progressing from procedural to conceptual knowledge.

Common Misconceptions

Students struggled with visualizing planes in 3D space and interpreting normal vectors. Confusion between vector and Cartesian equations of planes was prevalent. Students are struggling with 3D geometry concepts, particularly planes, normal vectors, and different equation forms. This typically falls within APOS levels as follows:

1. Action Level – Students can follow procedures to write equations of planes but struggle with interpretation.

2. Process Level – Students recognize connections between normal vectors and plane equations but may confuse Cartesian and vector forms.
3. Object Level – Students see plane equations as complete mathematical objects and can manipulate them flexibly.
4. Schema Level – Students integrate plane equations into broader 3D geometry concepts.

Based on this description, students are at the Action or early Process Level that is they can apply procedures but struggle with conceptual understanding and connections.

APOS Framework's Role

Most learners were at the Action and Process stages, with limited transition to Object and Schema.

Interactive and visual-based learning approaches were more effective in improving comprehension.

Interpretative Role of APOS Theory

APOS (Action, Process, Object, Schema) theory helps interpret how students conceptualize mathematical ideas. in this case, planes in 3D space. It explains why students may struggle with visualizing planes, normal vectors, and different equation forms by identifying their cognitive development stages:

Action Level – Students rely on rote procedures to manipulate plane equations without deeper understanding.

Process Level – They begin recognizing patterns and relationships, but misconceptions persist (e.g., confusing vector and Cartesian forms).

Object Level – Students can treat equations as unified objects and transition between forms fluidly.

Schema Level – They integrate plane equations into broader 3D geometry concepts (e.g., intersections, transformations).

In this case, most students are in Action and Process levels, meaning they can apply formulas but struggle to connect them conceptually.

Pedagogical and Instructional Contributions of APOS Theory

APOS provides a framework to design instructional strategies that help students transition through these stages:

1. Interactive and Visual-Based Learning
    - Dynamic geometry software (GeoGebra, Desmos 3D) to help visualize planes and normal vectors.
    - 3D models and augmented reality tools to support spatial reasoning.
    - Graphing calculators or physical manipulatives for hands-on exploration.
2. Bridging Action to Process
    - Encourage students to explain their reasoning verbally and in writing (e.g., why a normal vector defines a plane).
    - Use guided discovery activities that require students to compare and contrast vector vs. Cartesian forms.
    - Provide structured exercises transitioning from procedural to conceptual understanding (e.g., "What happens to the plane if the normal vector changes?").
3. Developing Object and Schema Thinking
    - Problem-solving tasks involving intersections of planes or transformations.
    - Real-world applications (e.g., engineering, physics problems) to integrate knowledge.
    - Collaborative learning through group discussions and explorations of multiple representations.

Since interactive and visual-based methods were more effective in the observations, these strategies align well with the students' current APOS levels

Effective Teaching Strategies

Conceptual teaching methods, including visualization tools and problem-based learning, improved students' understanding.

APOS-based instructional strategies yielded better cognitive progression than traditional teaching methods.

5.2.2 IMPLICATIONS

The study underscores the need for APOS-based instructional interventions to enhance vector comprehension.

Misconceptions in vector-plane concepts suggest a gap in curriculum design, requiring more emphasis on conceptual teaching.

Professional development for teachers is necessary to integrate cognitive-based teaching strategies effectively.

5.3 RECOMMENDATIONS

5.3.1 PEDAGOGICAL RECOMMENDATIONS

- ✓ Introduce interactive visual tools, such as 3D simulations, to enhance spatial reasoning.
- ✓ Implement APOS-based instructional approaches that encourage schema development through guided activities and exploratory tasks.
- ✓ Teachers should emphasize multiple representations (graphical, algebraic, and verbal) to reinforce conceptual clarity.

5.3.2 METHODOLOGICAL RECOMMENDATIONS

- ✓ Future research should adopt longitudinal studies to track conceptual development over time.
- ✓ Ensure a more engaging and comfortable research environment, possibly through peer-led discussions.

5.3.3 FUTURE RESEARCH DIRECTIONS

- Investigate the impact of technology-enhanced learning on students' comprehension of vectors in three-dimensional space.
- Explore how APOS theory can be integrated into teacher training programs for mathematics educators.

- Conduct comparative studies across different educational systems to identify best practices for teaching vectors and planes.

5.3.4 CHAPTER SUMMARY

This chapter has summarized the findings from chapter 1 which deals with problem statement, chapter 2 which deals on research gap found through the review of literature, chapter 3 which deals on sampling methods/techniques and data collection methods to be used and the reasons. This chapter has summarized key findings, drawn conclusions, and provided recommendations based on the study's outcomes. The insights gained contribute to enhancing mathematics instruction and addressing cognitive challenges in vector learning at the Advanced Level.

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APENDICES

QUESTIONNAIRE FOR LEARNERS

Questionnaire: Understanding of Vectors Involving Planes Concepts

For each statement below, please indicate whether you ***Agree, Disagree,*** or feel ***Neutral*** by using **a tick (√)**

STATEMENT	AGREE	DISAGREE	NEUTRAL
I am confident in identifying the components of a vector in three-dimensional space.			
I understand how to represent the position of a point in space using a position vector.			
I can find the equation of a plane if given a point on the plane and a normal vector to it.			
I can easily determine if two vectors in a plane are parallel or perpendicular.			
I understand the vector equation of a line that lies within a plane.			
I feel confident determining the line of intersection when two planes intersect.			
I am comfortable verifying if a given point lies on a specified plane.			
I understand the significance of a vector normal to a plane in describing the orientation of that plane.			
I feel confident finding the shortest distance from a point to a plane using vectors.			

I can think of real-world applications where knowledge of vectors involving planes is important.			

ADDITIONAL QUESTIONS FOR LEARNERS.

1. Action Level

Q1: Can you identify the components of a vector in three-dimensional space? Give an example and label each component.

Q2: How would you describe the position vector of a point in space? Provide an example with coordinates.

----

2. Process Level

Q3: Explain how you would find the equation of a plane given a point on the plane and a normal vector to the plane.

Q4: Given two vectors in a plane, how would you determine if they are parallel or perpendicular? Show the calculations involved.

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3. Object Level

Q5: Describe the vector equation of a line that lies within a plane. How would you relate this line to the plane?

Q6: If two planes intersect, explain how you would determine the line of intersection using vectors.

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4. Schema Level

Q7: What does it mean for a point to lie on a plane? Describe how you would verify if a given point lies on a specified plane.

Q8: In your own words, explain how the concept of a vector normal to a plane is useful in determining the orientation of that plane in space.

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5. Application and Conceptual Understanding

Q9: How would you approach solving a problem where you need to find the shortest distance from a point to a plane? Outline your process.

Q10: Describe a real-world situation where understanding vectors involving planes could be useful. How would you apply what you know in this context?

QUESTIONNAIRE FOR TEACHERS

Questionnaire: Teachers' Perceptions of Learners' Understanding of Vectors and Plane Concepts

For each statement below, please indicate whether you ***Agree, Disagree,*** or feel ***Neutral***

QUESTION/STATEMENT	**AGREE**	**DISAGREE**	**NEUTRAL**
Most learners in my class understand the basic components of a vector in three-dimensional space.			
Learners can confidently describe and use the concept of a position vector for points in space			
My students generally understand how to determine the equation of a plane given a point and a normal vector			
Learners find it straightforward to determine if two vectors are parallel or			
Most learners can construct and interpret the vector equation of a line lying within a plane.			
Learners understand how to find the line of intersection between two planes.			
My students are able to verify if a specific point lies on a given plane with confidence.			
Learners grasp the importance of a normal vector in describing a plane's orientation.			
Students find it challenging to calculate the shortest distance from a point to a plane.			

I believe there are effective real-world examples that help students understand vectors involving planes.			

INTERVIEW QUESTIONS FOR TEACHERS
UNDERSTANDING OF VECTORS AND PLANE CONCEPTS

- ✓ Could you describe your experience teaching vectors and plane concepts to advanced level students?
- ✓ How do you find students generally respond to these topics in terms of interest and engagement?
- ✓ In your experience, what are the most common misconceptions that students have about vectors and planes?
- ✓ Are there specific aspects of vectors involving planes that students find particularly challenging to understand? Could you elaborate?
- ✓ What teaching strategies or methods have you found most effective for explaining vectors and plane concepts?
- ✓ Do you use any particular visual aids, technology, or interactive activities to help students grasp these concepts?
- ✓ How do you assess whether students truly understand the concepts of vectors and planes, beyond rote memorization?
- ✓ Are there certain types of questions or problems that you use to gauge their understanding?
- ✓ Do you incorporate real-world applications of vectors involving planes in your lessons? If so, could you provide examples?
- ✓ How do you think applying vectors and planes to real-world contexts impacts student understanding and engagement?
- ✓ What are the main challenges you face in teaching vectors and planes to advanced level students?
- ✓ Are there any specific gaps in resources or support that you feel would help address these challenges?

- ✓ How would you describe your students' confidence in using vector and plane concepts? Do you notice changes in their confidence over time?
- ✓ What factors do you think contribute to building (or diminishing) student confidence with these topics?
- ✓ Have you participated in any professional development programs focused on teaching advanced mathematical concepts like vectors and planes?
- ✓ What additional training or resources do you feel would benefit you in teaching these topics effectively?
- ✓ Are you familiar with APOS (Action, Process, Object, Schema) theory, and if so, do you find it useful in your teaching of vector and plane concepts?
- ✓ Do you incorporate any elements of APOS theory in your instruction, either formally or informally? If yes, could you describe how?
- ✓ If you could suggest improvements to the curriculum or resources related to teaching vectors and planes, what would they be?
- ✓ In your view, what could be done at a school or system level to enhance students' understanding and performance in these areas?

PRE-TEST AND POST-TEST
VECTORS AND PLANE CONCEPTS

Instructions: Answer the following questions to the best of your ability. Show all workings where applicable. Total: $50\ marks$

Section 1: Basic Concepts

1. **Vector Components**
   - Define a vector in three-dimensional space and identify its components.
   - Example: Given vector **v** = (3, -2, 5), label each component.
2. **Position Vectors**
   - Explain what a position vector is and how it is used to represent a point in space.
   - Find the position vector of the point $P(4, -1, 3)$

Section 2: Vectors and Planes

3. **Equation of a Plane**
   - Given a point $A(1,2,3)$ on a plane and a normal vector **n** = (2, -1, 4), write the equation of the plane.
4. **Parallel and Perpendicular Vectors**
   - Given vectors **a** = (2, 4, 6) and **b** = (1, 2, 3), determine if **a** and **b** are parallel, perpendicular, or neither. Justify your answer.

Section 3: Lines and Planes

5. **Line in a Plane**
   - Write the vector equation of a line that passes through the point $B(1, -1, 2)$ and is parallel to the vector **d** = $(3, 2, -1)$

6. **Intersection of Two Planes**
   - Two planes have equations: Plane 1: $3x - y + 2z = 4$ and Plane 2: $x + y - z = 2$. Describe the method you would use to find the line of intersection.

Section 4: Application and Analysis

7. **Point on a Plane**
   - Given the plane equation $2x - 3y + z = 6$, determine if the point $C(1, -1, 2)$ lies on this plane.
8. **Distance from a Point to a Plane**
   - Calculate the shortest distance from the point $D(2, 3, -4)$ to the plane $x + 2y - z = 5$
9. **Normal Vector**
   - Explain the concept of a normal vector and its role in determining the orientation of a plane in three-dimensional space.
10. **Real-World Application**
    - Describe a real-world situation in which the concepts of vectors and planes are applied. How would understanding these concepts be useful in that scenario?

Scoring Guide:

- **Basic Concepts**: Questions 1–2 ($5\ marks$)
- **Vectors and Planes**: Questions 3–4 ($10\ marks$)
- **Lines and Planes**: Questions 5–6 ($15\ marks$)
- **Application and Analysis**: Questions 7–10 ($20\ marks$)