BINDURA UNIVERSITY OF SCIENCE EDUCATION SFM215

STOCHASTIC PROCESSES

BSc. STATISTICS AND FINANCIAL MATHEMATICS

Time : 3 hours NOV 2024

Candidates may attempt ALL questions in Section A and at most TWO questions in Section B. Each question should start on a fresh page.

SECTION A (40 marks)

Candidates may attempt ALL questions being careful to number them A1 to A5.

A1. Define the following terms.

(a) stochastic process, [2]

(b) accessible state, [2]

(c) counting process, [2]

(d) birth and death process, [2]

(e) hypo-exponential state, [2]

(f) memoryless property. [2]

A2. State and prove the Chapman Kolmogorov equations. [7]

A3. Specify the classes of the following Markov chains, and determine whether they are transient or recurrent:

$$P_1 = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

[6]

A4. Cars pass a certain street location according to a Poisson process with rate λ. A woman who wants to cross the street at that location waits until she can see that no cars will come by in the next T time units.

(a) Find the probability that her waiting time is 0. [5]

(b) Find her expected waiting time. [5]

Hint: Condition on the time of the first car.

A5. Mr. Kusotera works on a temporary basis. The mean length of each job he gets is three months. If the amount of time he spends between jobs is exponentially distributed with mean 2, then at what rate does Mr. Kusotera get new jobs? [5]

SECTION B (60 marks)

Candidates may attempt TWO questions being careful to number them B6 to B8.

B6. (a) Three white and three black balls are distributed in two urns in such a way that each contains three balls. We say that the system is in state $i, i = 0, 1, 2, 3$, if the first urn contains i white balls. At each step, we draw one ball from each urn and place the ball drawn from the first urn into the second, and conversely with the ball from the second urn. Let $X + n$ denote the state of the system after the nth step. Explain why $X_n, n = 0, 1, 2, ...$ is a Markov chain and calculate its transition probability matrix. [3, 5]

(b) Suppose that whether or not it rains today depends on previous weather conditions through the last three days. Show how this system may be analyzed by using a Markov chain. How many states are needed? [5]

(c) Each entering customer must be served first by server 1, then by server 2, and finally by server 3. The amount of time it takes to be served by server i is an exponential random variable with rate $\mu_i, i = 1, 2, 3$. Suppose you enter the system when it contains a single customer who is being served by server 3.

(i) Find the probability that server 3 will still be busy when you move over to server 2. [4]

(ii) Find the probability that server 3 will still be busy when you move over to server 3. [4]

(iii) Find the expected amount of time that you spend in the system. (Whenever you encounter a busy server, you must wait for the service in progress to end before you can enter service.) [5]

(iv) Suppose that you enter the system when it contains a single customer who is being served by server 2. Find the expected amount of time that you spend in the system. [4]

B7. (a) Consider a single-server bank for which customers arrive in accordance with a Poisson process with rate . If a customer will enter the bank only if the server is free when he arrives, and if the service time of a customer has the distribution G, then what proportion of time is the server busy? [6]

(b) Consider a train station to which customers arrive in accordance with a Poisson process having rate λ. A train is summoned whenever there are N customers waiting in the station, but it takes K units of time for the train to arrive at the station. When it arrives, it picks up all waiting customers. Assuming that the train station incurs a cost at a rate of nc per unit time whenever there are n customers present, find the long-run average cost. [8]

(c) Consider a semi-Markov process in which the amount of time that the process spends in each state before making a transition into a different state is exponentially distributed. What kind of process is this? [5]

(d) Potential customers arrive at a full-service, one-pump gas station at a Poisson rate of 20 cars per hour. However, customers will only enter the station for gas if there are no more than two cars (including the one currently being attended to at the pump. Suppose the amount of time required to service a car is exponentially distributed with a mean of five minutes.

(i) What fraction of the attendants time will be spent servicing cars? [6]

(ii) What fraction of potential customers are lost? [5]

B8. (a) Consider a queueing system having two servers and no queue. There are two types of customers. Type 1 customers arrive according to a Poisson process having rate λ_1, and will enter the system if either server is free. The service time of a type 1 customer is exponential with rate μ_1. Type 2 customers arrive according to a Poisson process having rate λ_2. A type 2 customer requires the simultaneous use of both servers; hence, a type 2 arrival will only enter the system if both servers are free. The time that it takes (the two servers) to serve a type 2 customer is exponential with rate μ_2. Once a service is completed on a customer, that customer departs the system.

(i) Define states to analyze the preceding model. [5]

(ii) Give the balance equations. [5]

(iii) In terms of the solution of the balance equations,

(a) find the average amount of time an entering customer spends in the system; [5]

(b) the fraction of served customers that are type 1. [5]

(b) For open queueing networks

(i) state and prove the equivalent of the arrival theorem; [6]

(ii) derive an expression for the average amount of time a customer spends waiting in queues. [5]

END OF QUESTION PAPER