BINDURA UNIVERSITY OF SCIENCE EDUCATION AMT113

BACHELOR OF APPLIED MATHEMATICS

LINEAR ALGEBRA II

Time : 3 hours

APR 2025

Candidates should attempt ALL questions in section A and at most TWO questions in section B.

SECTION A (40 marks)

Candidates may attempt ALL questions being careful to number them A1 to A6

A1. Define the following:

(a) Rank of a matrix. [1]

(b) Basis of a vector space. [2]

A2. (a) Define an inner product space. [2]

(b) State the axioms of an inner product space. [2]

(c) Let $\mathbb{R}^4$ have the Euclidean inner product. Find the cosine of the angle β between the vectors $u = (1, -1, 2, 0)$ and $v = (0, 3, -1, 4)$. [4]

A3. Find the basis and dimension of the subspace W of $\mathbb{R}^3$ given by:

$$W = \{(x, y, z) \in \mathbb{R}^3 : 2x - y + 3z = 0\}$$

[5]

A4. Use the Gram-Schmidt process to orthogonalize the vectors $v_1 = (2, 1, 0)$ and $v_2 = (1, -1, 1)$. [5]

A5. (a) Let $N = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 3 & 2 \\ 0 & 4 & 5 \end{pmatrix}$. Find:

(a) The row space of N. [2]

(b) The column space of N. [2]

(c) The null space of N. [3]

A6. (a) Define a linear combination. [2]

(b) Given the vectors $v_1 = (1, 2, -1)$ and $v_2 = (6, 4, 2)$. Show that $v_3 = (9, 2, 7)$ is a linear combination of v_1 and v_2. [4]

(c) Determine which of the following subsets of $\mathrm{Mat}_{3x3}(\mathbb{R})$ are linearly dependent. For those that are, express one vector as a linear combination of the others.

(i) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$. [2]

(ii) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. [2]

(iii) $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$. [2]

SECTION B (60 marks)

Candidates may attempt TWO questions being careful to number them B7 to B9.

B7. (a) Two matrices A and B are similar if there is an invertible matrix P such that $B = P^{-1}AP$. Prove that the similarity of matrices is an equivalence relation. [5]

(b) Let

$$C = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

Find the eigenvalues of C and a basis for the eigenspace of each eigenvalue of C. [12]

(c) Hence, or otherwise solve the system:

$$y_1' = 4y_1 + y_2$$

$$y_2' = 2y_1 + 3y_2$$

of differential equations involving functions y_1; y_2 of the variable t. [9]

(d) Express the quadratic form $x^2 + 4y^2 + 2xy + 6xz + 4zx$ in the form X^TAX [4]

B8. Let $A = \begin{pmatrix} 5 & 2 \\ 1 & 4 \end{pmatrix}$.

(a) Diagonalize the matrix A. [10]

(b) Solve the system of differential equations

$$\frac{dx}{dt} = 5x + 2y, \quad \frac{dy}{dt} = x + 4y$$

using matrix methods. [10]

(c) Prove the Cayley-Hamilton Theorem for A and use it to compute A^3 [10]

B9. (a) Define a span of a subspace X. [3]

(b) Determine whether the vectors $(2,1,0)$, $(1,-1,1)$, $(3,0,1)$ span the vector space $\mathbb{R}^3$. [10]

(c) Consider the transformation T that transforms 2×1 vectors into 3×1 vectors

$$T \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 2x + y \\ x - y \\ 4x + 3y \end{bmatrix}$$

Show that T is a linear transformation. [12]

(d) Let V be a subspace of $\mathbb{R}^4$ spanned by the vectors $x_1 = (1,1,1,1)$ and $x_2 = (1,0,3,0)$. Find an orthonormal basis of V. [5]

END OF QUESTION PAPER