BINDURA UNIVERSITY OF SCIENCE EDUCATION SFM123

BACHELOR OF STATISTICS AND FINANCIAL MATHEMATICS

PROBABILITY THEORY II



Time : 3 hours

Candidates should attempt ALL questions in section A and at most TWO questions in section B.

## SECTION A (40 marks)

Candidates may attempt ALL questions being careful to number them A1 to A5

**A1.** Show that

$$P\left(\lim_{n\to\infty}\{A_n\}\right) = \lim_{n\to\infty} P(A_n)$$

given that $\{A_n\}$ is a monotone increasing sequence of events. [5]

**A2.** Suppose that $X_1$ and $X_2$ are independent random variables each having the $\text{Exp}(\lambda)$ distribution. Determine the density functions of:

(a) $X = \max\{X_1, X_2\}$, [4]

(b) $Y = \min\{X_1, X_2\}$. [4]

**A3.** Consider an experiment of tossing two coins three times. Coin $A$ is fair, but coin $B$ is not fair, with $P(H) = \frac{1}{4}$ and $P(T) = \frac{3}{4}$. Consider a bivariate random variable $(X, Y)$, where $X$ denotes the number of heads resulting from coin $A$ and $Y$ denotes the number of heads resulting from coin $B$.

(a) Find the range of $(X, Y)$, [3]

(b) Find the joint probability mass function of $(X, Y)$, [4]

(c) Compute

(i) $P(X = Y)$, [2]

(ii) $P(X > Y)$, [2]

(iii) $P(X + Y \leq 4)$. [2]

**A4.** Let $X$ and $Y$ be independent random variables with $X$ being exponentially distributed with mean 1 and $Y$ being normally distributed in the interval $[0;1]$.
Find $\mathbb{E}[\max(X;Y)]$. [10]

**A5.** Let $X_1, ..., X_n$, be a random sample of $X$ with mean $\mu$ and variance $\sigma^2$. How many samples of $X$ should be taken if the probability that the sample mean will not deviate from the true mean $\mu$ by more than $\frac{\sigma}{10}$ is at least 0.95? [4]

## SECTION B (60 marks)

Candidates may attempt TWO questions being careful to number them B6 to B8.

**B6.** (a) Suppose that the joint probability density function of $X$ and $Y$ is given by:

$$f(x,y) = \begin{cases} 8xy, & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

(i) Verify that the $f(x,y)$ given above is indeed a p.d.f, [4]
(ii) Find the marginal probability density of $X$, $f_1(x)$, [4]
(iii) Find the marginal probability density of $Y$, $f_2(y)$, [4]
(iv) Are $X$ and $Y$ independent?, [4]
(v) Find the expected value of $X$. [4]

(b) Suppose that $X$ and $Y$ are independent standard normal random variables. Find the probability density function of $Z = X + Y$. [10]

**B7.** (a) Given that $\varphi_X(t)$ is the characteristic function of $X$, show that:

(i) $\varphi_{X+c}(t) = e^{itc}\varphi_X(t)$, [4]
(ii) $\varphi_{cX}(t) = \varphi_X(ct)$, [3]
(iii) $\varphi_{cX+d}(t) = e^{itd}\varphi_X(ct)$, [4]
(iv) $|\varphi_X(t)| \leq 1$. [4]

(b) Let $X \sim N(0,1)$. Find the characteristic function of $X$. [9]

(c) Let $X$ and $Y$ be two independent random variables, defined on the same probability space. What is the characteristic function of $X + Y$? [6]

**B8.** (a) Define the following modes of convergence:

(i) almost sure convergence,

(ii) convergence in probability,

(iii) convergence in distribution,

(iv) convergence in quadratic mean. [3,3,3,3]

(b) A company producing electric relays has three manufacturing plants producing 50, 30, and 20 percent, respectively of its product. Suppose that the probabilities that a relay manufactured by these plants is defective are 0.02, 0.05, and 0.01, respectively.

(i) If a relay is selected at random from the output of the company, what is the probability that it is defective? [3]

(ii) If a relay selected at random is found to be defective, what is the probability that it was manufactured by plant two? [4]

(c) Two numbers are chosen at random from among the numbers 1 to 10 without replacement. Find the probability that the second number chosen is 5. [5]

(d) Let $(X, Y)$ be a bivariate random variable. Show that $\mathbb{E}(X^2Y^2) \leq \mathbb{E}(X^2)\mathbb{E}(Y^2)$, the Cauchy-Schwatz inequality. [6]

**END OF QUESTION PAPER**