BINDURA UNIVERSITY OF SCIENCE EDUCATION MT504

MScED Mathematics

METRIC SPACES AND TOPOLOGY

JAN 2025

Time: 3 Hours

Candidates should attempt at most Four questions. Marks will be allocated as indicated.

A1. (a) Define the following terms,

(i) Equivalence relation

(ii) Partial order.

[2]

(b) Define R on $N \times N$ by $(a, b) \approx (c, d)$ if $ad = bc$.

(i) Prove that R is an equivalence relation. [8]

(ii) Let $A = \{1, 2, 3, \dots 15\}$. Let $\approx$ be the equivalence relation on $A \times A$ defined by $(a, b) \approx (c, d)$ if $ad = bc$. Find the equivalence class of (3,2). [4]

(c) Let X be a metric space with metric d. Show that,

$$d^1 = \frac{d(x, y)}{1 + d(x, y)}$$

is also a metric on X. [8]

(d) Let A and B be subsets of a metric space X. Show that, if $A \subset B$ then $\bar{A} \subset \bar{B}$. [3]

A2. (a) Prove that every Cauchy sequence is bounded. [4]

(b) (i) Define the term axiom of extensionality [2]

(ii) Show using extensionality that for all subsets of some universal set,

$$(A \cup B)^c = A^c \cap B^c.$$ [4]

(c) Let X = R^n or C^n for $x = (x_1, x_2, \dots, x_n)$, y = $(y_1, y_2, \dots, y_n)$. Define d : X × X → R^n by $d(x, y) = [\sum_{i=1}^{\infty} (x_i - y_i)^2]^{\frac{1}{2}}$. Prove that (X, d) is a metric space. [6]

(d) Is the space $C[-1, 1]$ complete with respect to the metric,

$$d(x, y) = \{\int_{-1}^{1} |x(t) - y(t)|^2 \, dt\}^{\frac{1}{2}}?$$

Justify your answer. [9]

A3. (a) Show that for any metric space (X, d),

$$|d(z,y) - d(x,y)| \leq d(x,z) \text{ for all } x, y, z \in X.$$ [4]

(b) Let R be the relation in **R** defined by $x{\sim}y$ if and only if $x - y$ is an integer. Prove that ~ is an equivalence relation. [6]

(c) Let X be a metric space. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $x_n \to x$ and $y_n \to y$, show that $d(x_n, y_n) \to d(x, y)$. [7]

(d) Prove the following property of the Euclidean Norm: $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$. [6]

(e) The set $\boldsymbol{R}$ of real numbers is not separable. Justify whether the statement is true or false. [2]

A4. (a) Let T_1 and T_2 be two topologies on a non-empty set X. Show that $T_1 \cap T_2$ is also a topology on X. [8]

(b) Show that the union of two topologies is not necessarily a topology. [6]

(c) Let $X = \{a, b, c, d\}$. Determine whether or not each of the following classes of subsets of X is a topology on X,

$$T_1 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$$

$$T_2 = \{X, \emptyset, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$$

$$T_3 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c, d\}\{a, b, c, d\}\}.$$

[11]

A5. (a) Let X be a metric space. Prove that if G_1 and G_2 are open in X, then $G_1 \cap G_2$ is also open in X. [7]

(b) Prove that the space l^p, where $1 \leq p \leq \infty$ is complete. [8]

(c) Let (X, d) and (Y, ρ) be metric spaces and $f: X \to Y$. Prove that the following statements are equivalent,

(i) f is continuous on X.

(ii) for any open set $G \subset Y$, $f^{-1}(G)$ is open in Y.

(iii) for any closed set F in Y, $f^{-1}(F)$ is closed in X. [10]

A6. (a) Define a contraction on a metric space (X, d). [3]

(b) State and prove the contraction mapping theory. [9]

(c) Consider the metric space $\boldsymbol{R}$, with the usual metric and define $f: \boldsymbol{R} \to \boldsymbol{R}$ by

$f(x) = (1 + x)^{\frac{1}{3}}$.

(i) Show that $f(x)$ is a contraction on [1, 2]. [4]

(ii) Using an initial guess of $x_0 = 1$, find the fixed point of $f(x)$ correct to 3 decimal places. [9]

END OF QUESTION PAPER