BINDURA UNIVERSITY OF SCIENCE EDUCATION MT112

Faculty of Science & Engineering

MATHEMATICAL DISCOURSE AND STRUCTURES

Time : 3 hours

- JUN 2023

Candidates may attempt ALL questions in Section A and at most TWO questions in Section B. Each question should start on a fresh page.

SECTION A (40 marks)

Candidates may attempt ALL questions being careful to number them A1 to A6.

A1. We write $P \downarrow Q$ as an abbreviation for $\neg(P \wedge Q)$. This connective is called a Sheffer stroke. Use the truth table to verify the following logical equivalence.

$$P \wedge Q \equiv (P \downarrow P) \downarrow (Q \downarrow Q).$$

[4]

A2. (a) Prove that if A and B are non-empty sets, then $A \times B = B \times A$ iff $A = B$. [5]

(b) Let R be a relation on $\mathbb{Z}$ defined by: mRn iff 5 divides $m - n$. Show that R is an equivalence relation. [6]

A3. (a) Let S be any subset of real numbers. Show that the relation in S defined by $x \leq y$ is a partial order in S. [3]

(b) Let $f : A \to B$ be a function. Define the following.

(i) f is surjective. [1]

(ii) f is injective. [1]

(iii) When is f said to be bijective? [1]

A4. (a) If A and B are subsets of set X, and $A - B = \{a \in A \mid a \notin B\}$, prove that $A - B = A \cap B'$ where $B' = X - B$ is the complement of B in X, i.e. $B' = \{x \in X \mid x \notin B\}$. [3]

(b) Let $\{A_i\}_{i \in I}$ be an indexed family of sets and B be any set, then show that $B - (\cup_{i \in I} A_i) = \cap_{i \in I}(B - A_i)$. [4]

A5. Let $*$ be a binary operation on $\mathbb{Z}$ defined for $\forall\ m, n \in \mathbb{Z} : m * n = (m+n)^2$.

(a) Discuss whether $*$ is commutative and associative. [3]

(b) Show whether $\mathbb{Z}$ is idempotent with respect to $*$. [4]

A6. Use mathematical induction to prove the following statement for every $n \in \mathbb{N}$.

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

[5]

SECTION B (60 marks)

Candidates may attempt TWO questions being careful to number them B7 to B9.

B7. (a) Show using extensionality that for all subsets of some universal set

$$(A \cup B)' = A' \cap B'.$$

[4]

(b) The standard definition of a real sequence $\{S_n\}$ being convergent to a limit l is

$$\forall \varepsilon > 0\ \exists\ n_0 \in \mathbb{N};\ \forall n > n_0,\ \mid S_n - l \mid < \varepsilon.$$

Write down the formal definition of $\{S_n\}$ not converging to l. [4]

(c) (i) Define a group. [4]

(ii) Find out if the mathematical structure $(P(X), \cap, \cup)$ is a ring or not for any non-empty set X. [7]

(d) (i) Give the definition of a monoid. [3]

(ii) Define the operation on $\mathbb{R}$ by $a \triangle b = a + b + ab$. Show that the binary operation makes $\mathbb{R}$ into a monoid. [4]

(iii) Find the invertible elements of R with respect to $\triangle$. [4]

B8. (a) Let A be a set of integers, and let $\sim$ be the relation on $A \times A$ defined by $(a, b) \sim (c, d)$ iff $a + d = b + c$.

(i) Prove that $\sim$ is an equivalence relation. [7]

(ii) Suppose $A = \{1, 2, 3 \cdots 8, 9\}$. Find [(2,5)], the equivalence class of (2,5). [2]

(b) Consider the functions $f : A \to B$ and $g : B \to C$. Prove the following.

(i) If $g \circ f$ is one to one, then f is one to one. [3]

(ii) If $g \circ f$ is onto, then g is onto. [4]

(c) If A, B and C are non-empty sets show that

$$(A \cup B) \times C = (A \times C) \cup (B \times C),$$

using axioms of extensionality. [5]

(d) Prove that, for any integers a and b the product ab is even if and only if a is even or b is even. [4]

(e) Use the method of proof by contradiction to show that if $x \in \mathbb{R}$ and $x^2 = 2$, then $x \notin \mathbb{Q}$. [5]

B9. (a) Distinguish between a tautology and a contradiction. [2]

(b) The following conditionals and bi-conditionals are tautologies. Construct truth tables to verify each of the laws.

(i) $((p \Rightarrow q) \wedge p) \Rightarrow q$. [3]

(ii) $[(p \vee q) \wedge (\neg p)] \Rightarrow q$ (Disjunctive syllogism). [5]

(c) (i) Test the validity of the following: If I study then I will not fail mathematics. If I do not play basketball then I will study. However I failed mathematics therefore I must have played basketball. [6]

(ii) Prove whether the following argument is valid or not.

$$p \rightarrow \neg q, \quad r \rightarrow q, \quad r \vdash p \neg p.$$

[4]

(d) Let $A = \{1, 2, 3, 4, 5\}$. Write down the negation of each of the following statements, the truth values of the statements and the negation. Find a counter example for each false statement or false negation.

(i) $\exists x \in A: \quad x + 3 = 10,$ [1]

(ii) $\forall x \in A: \quad x + 3 < 10,$ [2]

(iii) $\forall x \in \{2, 3, \cdots, 9\}, \; x + 5 < 12.$ [2]

(e) Prove that for each positive integer n, $\; n! > 2^n$ for $n \geq 4$. [5]

END OF QUESTION PAPER