BINDURA UNIVERSITY OF SCIENCE EDUCATION MT513

MScED Mathematics

FUNCTIONAL ANALYSIS JAN [illegible]5

Time: 3 Hours

Candidates should attempt at most Four questions. Marks will be allocated as indicated.

A1. (a) Define the following terms

(i) a metric space on a set X. [2]

(ii) Cauchy sequence. [1]

(iii) l^p space. [2]

(b) Consider the space (X, ρ) where $\rho(x, y) = \sqrt{|x - y|}$. Prove that (X, ρ) is a metric space. [8]

(c) Define R on $N \times N$ by $(a, b) \approx (c, d)$ if $ad = bc$.

(i) Prove that R is an equivalence relation. [8]

(ii) Let $A = \{1, 2, 3, \ldots 15\}$. Let $\approx$ be the equivalence relation on $A \times A$ defined by $(a, b) \approx (c, d)$ if $ad = bc$. Find the equivalence class of (3,2). [4]

A2. (a) Let V be a vector space over R. Define the inner product $<.,.>$ on V. [5]

(b) Prove that every inner product space is a normed linear space. [10]

(c) State and prove the Cauchy-Bunyakovsky theory. [10]

A3. (a) State and prove Holder inequality. [9]

(b) Prove that the open ball $B(x_0, \varepsilon)$ is an open set. [6]

(c) Is the space $C[-1, 1]$ complete with respect to the metric

$$d(x, y) = \{\int_{-1}^{1} |x(t) - y(t)|^2 \, dt\}^{\frac{1}{2}}?$$

Justify your answer. [10]

A4. (a) State without proof the characterisation of best approximation theory. [2]

(b) Let X, Y be normed linear spaces over F and $T \in B(X, Y)$, then prove that

$$\|T\| = Sup \left\{ \frac{\|T_x\|}{\|T\|} \right\}, x \in X \setminus 0.$$

[7]

(c) If a sequence (X_n) in an inner product space converges to x, then it converges weakly. [7]

(d) Prove that the Euclidean n-space R^n is complete. [9]

A5. (a) For any sets A and B prove that

(i) $(A - B) \cap B = \emptyset$. [4]

(ii) $(A \cap B)^c = A^c \cup B^c$. [6]

(b) For any give sets M and N show that $M \times N = \emptyset$, if and only if $M = \emptyset$ and $N = \emptyset$. [8]

(c) If T_n is a sequence of bounded linear operators from a Banach space X into a normed linear space Y. Prove that if T_n is strongly convergent to an operator T, then T is a bounded linear operator. [7]

A6. (a) Let R be the relation in **R** defined by $x \sim y$ if and only if $x - y$ is an integer. Prove that $\sim$ is an equivalence relation. [7]

(b) Prove that the space $C[a, b]$ define a metric space. [8]

(c) Let H be a Hilbert space and M a closed subspace of H. For each $x \in H/M$, show that there is a unique element $y_0 \in M$: $||x - y_0|| = inf_{y \in M} ||x - y||$. [10]

END OF QUESTION PAPER