BINDURA UNIVERSITY OF SCIENCE EDUCATION MT506

MScEd

NUMERICAL METHODS FOR PDEs

Time : 3 hours AUG 2024

Candidates may attempt at most FOUR questions. Full marks can be obtained for complete solutions to FOUR questions. Each question should start on a fresh page.

A1. Consider the following finite difference methods to integrate the heat equation $u_t = u_{xx}$. Here $\mu = \frac{k}{h^2}$ where k denotes the time step and h the space step. In each find the order of accuracy and use von Neumann analysis to determine the range μ for which the method is stable.

(a) Forward Euler scheme, [10]

(b) Crank-Nicholson method. [15]

A2. Consider the advection diffusion equation $u_t + au_x - u_{xx} = 0$ with Dirichlet boundary conditions and $a \geq 0$. Show that $u(x,t) = exp[-(iL\pi a + L^2\pi^2) + iL\pi^2]$ is for every L a particular solution of the problem. Use von Neumann analysis to derive the the stability conditions for the upwind scheme

$$\frac{u_j^{n+1} - u_j^n}{k} + a\frac{u_j^{n+1} - u_{j-1}^{n+1}}{h} = \frac{u_{j+1}^{n-1} - 2u_j^{n+1} + u_{j-1}^{n-1}}{h}$$

What is the degree of accuracy of the scheme? [25]

A3. (a) Show that the scheme of the form, $v_m^{n+1} = \alpha v_{m+1}^n + \beta v_{m-1}^n$, are stable if $|\alpha| + |\beta|$ is less than or equal to 1. [12]

(b) Show that the Lax-Friedrichs scheme $\dfrac{v_m^{n+1} - \frac{1}{2}\left(v_{m+1}^n + v_{m-1}^n\right)}{k} + a\dfrac{v_{m+1}^n - v_{m-1}^n}{2h} = 0$ is stable if $|a\lambda| < 1$. [13]

A4. (a) Show that the following modified Lax-Friedrichs scheme for the one way wave equation $u_t + au_x = f$, given by
$v_m^{n+1} = \dfrac{1}{2}\left(v_{m+1}^n + v_{m-1}^n\right) - \dfrac{a\lambda}{1+(a\lambda)^2}\dfrac{v_{m+1}^n - v_{m-1}^n}{2h} + kf_m^n$ is stable for all the values of λ. Discuss the relation of this explicit and unconditional stable scheme to the theorem which states that There are no explicit unconditionally stable, consistent finite difference schemes for hyperbolic systems of partial differential equations. [13]

(b) Consider the parabolic heat equation $u_t = bu_{xx}$, where b is a positive number. For the scheme

$$v_m^{n+\frac{1}{4}} = \frac{1}{4}kb\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2}$$

$$v_m^{n+1} = v_m^n + kb\frac{v_{m+1}^{n+\frac{1}{4}} - 2v_m^{n+\frac{1}{4}} + v_{m-1}^{n+\frac{1}{4}}}{h^2}$$

is stable for all the values of λ. Discuss the relation of this explicit and unconditional stable scheme to the theorem which states that There are no explicit unconditionally stable, consistent finite difference schemes for hyperbolic systems of partial differential equations. [12]

A5. Define the following terms

(i) Consistency, [2]

(ii) Stability, [4]

(iii) Convergence. [4]

Consider the one way wave equation $u_t + au_x = 0$ where a is a constant.

(i) In one paragraph describe what you understand by hyperbolic equations. [5]

(ii) Show that the forward time central space (FTCS) scheme given by $\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^n - v_{m-}^n}{2h} = 0$ is consistent with the one way wave equation. [5]

(iii) Show that the leap-frog scheme $\frac{v_m^{n+1} - v_m^n}{2k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0$ is consistent with the one way wave equation. [5]