BINDURA UNIVERSITY OF SCIENCE EDUCATION MT311

BACHELOR OF SCIENCE EDUCATION

ANALYSIS 11

AUG 2023

Time: 3 Hours

Candidates may attempt ALL questions in Section A and at most TWO questions in Section B.

Each question should start on a fresh page.

SECTION A (40 marks)

A1. Let A be a given subset of $\Re^n$. Define the following;

(a) open ball, [1]

(b) closed ball, [1]

(c) sphere. [1]

A2. If x and y are any two vectors in $\Re^n$, then prove that

$$|< x\,,\ y >| \leq \|x\|\|y\|, x.y \in \Re^n.$$ [8]

A3. (a) Using an $\varepsilon - \delta$ argument, show that

$$lim_{(x,y)\to(0,0)} \frac{y^3}{x^2 + y^2} = 0.$$

[5]

(b) Given the function

$$f(x,y) = \frac{2xy}{x^2 + y^2}, (x,y) \neq (0,0).$$

Show that the limit $f(x,y)$ does not exist about the point (0,0). [5]

A4. Find all the local maxima and the local minima of the function

$$z = f(x,y) = 6x^2 - 2x^3 + 3y^2 + 6xy.$$

[8]

A5. (a) Define the term metric space. [3]

(b) Prove that the function $d(x,y) = \|x - y\|$ for $(x,y) \in \boldsymbol{R}^n$ is a metric on $\boldsymbol{R}^n$. [5]

A6. Prove that the limit of a convergence sequence is unique. [3]

SECTION B (60 marks)

Candidates may attempt TWO questions being careful to number them B7 to B9.

B7. (a) Let X be a metric space on $\boldsymbol{R}^n$, prove that if G_1 and G_2 are open then $G_1 \cap G_2$ is open. [6]

(b) Use appropriate definition to show that

(i) $< x + y, z > = < x, z > + < y, z >$, [4]

(ii) $< ax, by > = ab < x, y >$. [3]

(c) Prove the following triangular inequality $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$. [6]

(d) Let (X, d) be a metric space. Define $\rho: X \times X \to \mathfrak{R}$ by

$$\rho(x, y) = \frac{d(x, y)}{1 + (d, y)}$$

$\forall\ (x, y) \in X$. Show that ρ is a metric on X. [8]

(e) Prove that $A \subset \mathfrak{R}^n$ is open if and only if $A = A^0$. [3]

B8. (a) Show that the improper integral of the first kind

$$\int_1^\infty \frac{1}{x^p} dx$$

converges if $p > 1$ and diverges if $p \leq 1$. [6]

(b) Test the following for uniform convergence

$$\sum_{k=1}^{\infty} \frac{sin2kx}{(2k+1)^{\frac{3}{2}}}.$$

[4]

(c) Determine whether the following improper integral converges or diverges

$$\int_1^\infty \frac{3x^2 + 2x - 1}{4x^4 + 3x^2 + 1} dx.$$

[5]

(d) Find all the local maxima and the local minima of the function

$x^3 - 3x^2 + y^2$. [8]

(e) Prove that if a function f of two variables is differentiable at (x_0, y_0) then f is continuous at (x_0, y_0). [7]

B9. (a) (i) State without proof Green's theorem. [3]

(ii) Use Green's theory to evaluate the line integral

$$\oint (-2xy + y^2)dx + x^2 dy$$

where C is the boundary of the region R enclosed by $y = 4x$ and $y = 2x^2$. [6]

(b) Evaluate

$$\iint 3xy^2$$

if the region R is enclosed by $y = x^2$ and $y = 2x$. [6]

(c) Evaluate

$$\iint e^{x^2+y^2}\, dxdy$$

where R is the region in the first quadrant inside the circle $x^2 + y^2 = a^2$. [5]

(d) (i) State the inverse function theorem. [4]

(ii) Given the vector function $f: \Re^2 \to \Re^2$, $f(x, y) = (u, v)$ where $u = 2x - 3y$ and $v = x + 2y$. Verify that the inverse function theorem is applicable and find the inverse function g. [6]

END OF QUESTION PAPER