**BINDURA UNIVERSITY OF SCIENCE EDUCATION** **MT502**

**MSc. Education: Part 1**

AUG 2024

**Non-Linear Differential Equations**

**Time: 3 hours**

Candidates may attempt at most FOUR questions. Full marks can be obtained for complete solutions to FOUR questions. Each question should start on a fresh page.

**Q1.** (a) Suppose we have the differential equations:

$$\frac{dx}{dt} = x(1 - x - y)$$

$$\frac{dy}{dt} = y(\frac{1}{2} - \frac{1}{4}x - \frac{3}{4}y)$$

Let $x$ and $y$ represent population densities of two species of bacteria competing for food supply.

(i) Examine whether there are equilibrium states that might be reached. [4]

(ii) Examine whether a periodic growth and decay will be observed. [3]

(iii) How such possibilities depend on the initial state of the two species? [8]

(b) Solve $X' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} X$. [10]

**Q2.** (a) State the Routh-Hurwitz criteria for stability of a syste of ordinary equation. [5]

(b) Investigate the stability of the zero solution of the characteristic equation $r^4 + 2r^3 + 4r^2 + 7r + 3 = 0.$ [7]

(c) Consider a non-linear first order logistic equation

$$\frac{dy}{dt} = f(y), \quad f(y) = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)y, \quad where\ r > 0\ and\ 0 < T < K.$$

(i) Sketch f(y) verse y. [3]

(ii) Obtain the critical points and discuss qualitatively their stability. [6]

(iii) Sketch qualitatively the graph of y($t$) verse $t$, and discuss the behaviour of y($t$) as $t \to \infty$ and $0 < y(0) < T$ and $T < y(0) < K$. [4]

**Q3.** The following equation models the growth of a population which is subject to predation

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - f(x), \qquad x \geq 0$$

where $f(x) = \frac{AX}{x+B}$ and $r$, $k$ and $B$ are strictly positive and $A$ is non-negative.

(a) (i) Show that the scaled variable $u = \frac{x}{B}, r = \frac{A}{B}t$ can be chosen so that the equation becomes:

$$\frac{du}{dr} = pu(q-u) - \frac{u}{u+1}. \qquad (1)$$

(ii) What is $p$ and $q$ in terms of $r, k, A$, and $B$? [4]

(b) Sketch bifurcations diagrams for the equilibrium point of equation (1) as the parameter p varies with q fixed in the two cases

(i) $q < 1$ and

(ii) $q > 1$

Your diagrams should show which of the equilibrium solutions are stable and which are unstable. [13]

(c) Use the result above to determine the behavior of the population $x$ as the parameter A (proportional to the size of the predator population) and increases slowly from 0 to very large values in the two cases:

(i) $k < B$ and

(ii) $k > B$.

What happens if A is decreased slowly back to 0 again. Can hysteresis occur? [8]

**Q4.** (a) Solve the system $X' = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} X + \begin{pmatrix} t^{-1} \\ 2t^{-1} + 4 \end{pmatrix}$. [10]

(b) Show that the equation

$$\ddot{x} + (x^2 - \mu)\dot{x} + 2x + x^3 = 0$$

has a bifurcation point at $\mu = 0$ and is oscillatory for some $\mu > 0$. [9]

(c) Determine the critical point $(x_0, y_0)$ and then classify its type and examine its stability by marking the transformations

$$x = x_0 + u$$
$$y = y_0 + v$$

for the following equation:

$$\frac{dx}{dt} =- 1 - x - y,$$

$$\frac{dy}{dt} = 5 + 2x - y. \qquad [6]$$

**Q5.** (a) Let $A = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$, find the Jordan form of A. [3]

(b) (i) State the Liapunov stability theorem. [3]

(ii) State when do we say a system is a strong Liapunov function. [3]

(iii) Show that the function $V(y_1, y_2) = y_1^2 + y_1^2 y_2^2 + y_2^4$, $(y_1, y_2) \in \mathbb{R}^2$ is a strong Liapunov function for a system:

$$\dot{x_1} = 1 - 3x_1 + 3x_1^2 + 2x_2^2 - x_1^3 - 2x_1x_2^2$$

$$\dot{x_2} = x_2 - 2x_1x_2 + x_1^2x_2 - x_2^3$$

at a fixed point. [7]

(c) (i) State without proof the linearization theorem. [3]

(ii) Show that the system

$$\dot{x_1} = e^{x_1+x_2} - x_2$$

$$\dot{x_2} = -x_1 + x_1x_2$$

have only one fixed point. Find the linearisation of this point. [3,3]

**Q6**. (a) Which of the following equation is autonomous or non-autonomous?

(i) $X'(t) = coste^{xt}$,

(ii) $X'(t) = e^{x(t)}$,

(iii) $X'(t) = X(t)(e^{6t})$. [1,1,1]

(b) Suppose function F: $R^2 \to R$ has the property that $F(\tau; \varphi) = F(-\tau; -\varphi)$ for all $(\tau; \varphi) \in R^2$. Prove that if $X: R \to R$ is a solution to the differential equation $X'(t) = F(t, X(t))$, then so is Y where $Y(t) =- X(-t);\ t \in R$. [4]

(c) Find the $2 \times 2$ matrix A such that the system $X'(t) = AX(t)$ has a solution

$$X(t) = \begin{pmatrix} e^{-t}(\cos t + 2\sin t) \\ e^{-t}\cos t \end{pmatrix}, t \in R.$$ [6]

(d) Consider the I-dimensional differential equation

$$x'(t) = (x(t))^2 - 5ax(t) + 6a^2, \text{ where } a \text{ is a real constant.}$$

(i) Find the equilibrium points of the equation. [3]

(ii) The equation has qualitatively different phase portraits, depending on $a$. Sketch the phase portraits for the equation where $a > 0$, when $a = 0$, and when $a < 0$. [9]

**END OF PAPER**