BINDURA UNIVERSITY OF SCIENCE EDUCATION MT108

BACHELOR OF SCIENCE EDUCATION

LINEAR ALGEBRA

– JAN 2025

Time: 3 Hours

Candidates may attempt ALL questions in Section A and at most TWO questions in Section B.

Each question should start on a fresh page.

SECTION A (40 marks)

A1. Define the following terms:

(i) Basis of a vector space. [1]

(ii) Subspace of a vector space. [2]

A2. Prove that if $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in a real inner product space then,

$$\|\boldsymbol{u}+\boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|.$$ [6]

A3. Determine whether or not the vectors $\boldsymbol{u_1}$ =(1, 2, -3), $\boldsymbol{u_2}$ = (1, -3; 2) and u_3 = (2, -1, 5) form a linearly independent set. [6]

A4. Let $A = \begin{bmatrix} 1 & -2 & 4 \\ -4 & 0 & 3 \\ 2 & -1 & 0 \end{bmatrix}$.

(a) Find the adjoint of the matrix A. [5]

(b) Hence or otherwise find A^{-1}, the inverse of matrix A. [2]

A5. (a) Find the solution of the following system of linear equations using Gauss elimination method

$$2x_1 - x_2 + x_3 = 4$$
$$-3x_1 + 2x_2 - 4x_3 = 1$$
$$x_1 - 5x_3 = 0 .$$ [6]

(b) Find a non zero matrix $B \neq I_2$ such that $AB = BA$ where $A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$. [4]

A6. (a) State the Cayley Hamilton theorem

(b) Verify the Cayley Hamilton theorem using $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. [4]

(c) Hence find A^2 and A^3 and A^n using the Cayley Hamilton theorem. [4]

SECTION B (60 marks)

Candidates may attempt TWO questions being careful to number them B7 to B9.

B7. (a) Find the solution to the linear system of equations,

$$5x_1 + x_2 - x_3 = 4$$
$$9x_1 + x_2 - x_3 = 1$$
$$x_1 + 2x_2 + 3x_3 = 11$$

using Crammers rule. [8]

(b) Find the values of k for which the following system of linear equations,

$$2x_1 + x_2 = 5$$
$$x_1 - 3x_2 = -1$$
$$3x_1 + 4x_2 = k$$ is consistent. [4]

(c) State the difference between rank of a matrix and the trace of a matrix. [2]

(d) Let $A = \begin{bmatrix} 1 & 3 & 0 \\ 4 & -1 & 2 \end{bmatrix}$, calculate A^tA. [4]

(e) Solve the following system of simultaneous equations using Gauss elimination.

$$x_1 - 3x_2 + 2x_3 + x_4 = -4$$
$$2x_1 - 6x_2 + x_3 + 4x_4 = 1$$
$$x_1 - x_2 - x_3 - 2x_4 = 12$$
$$-x_2 + x_3 + x_4 = 0.$$ [12]

B8. (a) Find the equation of a plane containing the points: P(2, 1, 1), Q(0, 4, 1) and R(-2, 1, 4). [6]

(b) Find the volume of the parallelepiped formed by the vectors $\boldsymbol{u} = <1, 2, 4>, \boldsymbol{v} = <2, 4, 1)$ and $\boldsymbol{w} = <5, 1, 0>$. [5]

(c) For any vectors **a** and **b** prove the identity $|\mathbf{a} \times \boldsymbol{b}|^2 + |\boldsymbol{a}.\boldsymbol{b}|^2 = |\boldsymbol{a}|^2|\boldsymbol{b}|^2$. [5]

(d) Find the parametric equation of a line in space which passes through the points P(1; 0; −1) and Q(3; −2; −3). [3]

(e) Given that $\boldsymbol{a} = 3\boldsymbol{i} - 2\boldsymbol{j} + 6\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k}$. *Find* $\boldsymbol{a} \times \boldsymbol{b}$ and a unit vector perpendicular to the plane containing the vectors **a** and **b**. [4]

(f)Given the matrix

$$C = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}.$$

Find the eigenvalues of C and a basis of each eigenspace of C. [7]

B9. (a) (i) Define the term linear independence of a set of vectors. [2]

(ii) Determine whether or not the vectors (1; −2; 1); (2; 1; −1) and (7; −4; 1) are linearly independent. [5]

(b) Let $\boldsymbol{u} = (u_1;\ u_2)$ and $\boldsymbol{v} = (v_1;\ v_2)$ be vectors in $\boldsymbol{R}^2$. Show that the Weighted Euclidean inner product $< \boldsymbol{u};\ \boldsymbol{v} >= 3u_1v_1 + 2u_2v_2$ satisfies all the inner product. [6]

(c) Find a basis and the dimension of the subspace W of R^4 spanned by $u_1 = (1;\ -2;\ 5;\ -3)$, $u_2 = (2; 3; 1; -4)$, and $u_3 = (3;\ 8;\ -3;\ -5)$. [5]

(d) Show that W is not a subspace of V= $\boldsymbol{R}^3$ where W consists of those vectors whose length does not exceed 1, i.e $W = \{(a, b, c): a^2 + b^2 + c^2 \leq 1\}$. [5]

(e) (i) Define the term linear combination of vectors. [1]

(ii) Express $A = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$ as a linear combination of the matrices $X = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $Z = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}$. [6]

END OF QUESTION PAPER