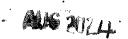
## MScEd Mathematics

## FUNCTIONAL ANALYSIS



Time: 3 hours

Candidates should attempt at most FOUR questions. Marks will be allocated as indicated

- A1. (a) Define the following terms
  - (i) a metric space.

[2]

(ii) the  $l^p$  space.

[2]

[9]

(iii) a functional on a linear space X.

[3]

(b) Consider  $F: X \to Y$  be an arbitary mapping. Define a relation R such that  $x_1 \sim x_2 \Rightarrow f(x_1) = f(x_2)$ . Prove that  $\sim$  is an equivalence relation.

 $x_1 \sim x_2 \Rightarrow f(x_1) = f(x_2)$ . Prove that  $\sim$  is an equivalence relation. (c) If (X, d) is any metric space, show that another metric on X is defined by

$$\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}.$$
 [9]

**A2.** (a) Let X be a linear space over F. Define an inner product on X.

(b) Let  $x, y, z \in X$  be an inner product space and  $\lambda \in F$ . Show that  $\langle x, 0 \rangle = 0 \ \forall x \in x$ .

[5]

[4]

(c) Let (X, <.,.>) be an inner product over F. Prove that  $\forall x, y \in X$ ,

$$| < x, y > | \le \sqrt{< x, x > < y, y >}.$$

[8]

(d) Prove that every inner product space is a normed linear space.

A3. (a) Define the term weak convergence of a sequence.

[3]

(b) Let  $(X, \| . \|)$  be a normed linear space. For  $x, y \in X$  define  $d(x, y) = \| x - y \|$ . Prove that d defines a metric on X.

(c) Let T be a linear operator from a normed linear space X into a normed linear space Y. Prove that the following statements are equivalent,

(i) T is continuous on X.

[2]

(ii) T is continuous on 0.

[3]

(iii) T is bounded.

[3]

- (d) Let X and Y be normed linear spaces over F. Prove that B(X,Y) is a normed linear space over F. [8]
- A4. (a) State and prove the projection theorem.

[6]

(b) Prove that the open ball  $B(x_0, \epsilon)$  is an open set.

[7]

- (c) Let R be the relation in the set of real numbers defined by aRb if  $0 \le a b \le 1$ . Express R and  $R^{-1}$  as subsets of  $\mathbf{R} \times \mathbf{R}$  and graph the relations. [12]
- **A5.** (a) Define the adjoint of an operator  $T: X \to Y$  where X and Y are normed linear spaces. [3]
  - (b) Prove that if p > 1 and q is such that  $\frac{1}{p} + \frac{1}{q} = 1$ , then  $a^{\frac{1}{p}} \cdot b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}$ , where  $a, b \in \mathbb{R}^+$ .
  - (c) State and prove Minkowski inequality.

[7]

(d) Is the space C[-1,1] complete with respect to the metric

$$d(x,y) = \left\{ \int_{-1}^{1} |x(t) - y(t)|^2 dt \right\}^{\frac{1}{2}}?$$

Justify your answer.

[8]

- **A6.** (a) Let H be a Hilbert space and M a closed subspace of H. For each  $x \in H$  minus M, there is a unique element  $y_0 \in M : ||x y_0|| = \inf_{y \in m} ||x y||$ . [15]
  - (b) Let  $T_n$  be a sequence in B(X,Y) where X is a Banach space. Assume that the sequence ( $||T_nx||$ ) is bounded for each  $x \in X$  i.e  $\forall x \in X \exists C_x > 0 : ||T_nx|| \leq C_x \forall n$ . Prove that  $\exists m > 0 : ||T_n|| \leq m$ . [10]