BINDURA UNIVERSITY OF SCIENCE EDUCATION MT513

MScEd Mathematics

FUNCTIONAL ANALYSIS

AUG 2024

Time : 3 hours

Candidates should attempt at most FOUR questions. Marks will be allocated as indicated .

A1. (a) Define the following terms

(i) a metric space. [2]

(ii) the l^p space. [2]

(iii) a functional on a linear space X. [3]

(b) Consider $F : X \to Y$ be an arbitary mapping. Define a relation R such that $x_1 \sim x_2 \Rightarrow f(x_1) = f(x_2)$. Prove that $\sim$ is an equivalence relation. [9]

(c) If $(X, d)$is any metric space, show that another metric on X is defined by $\rho(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$. [9]

A2. (a) Let X be a linear space over F. Define an inner product on X. [4]

(b) Let $x, y, z \in X$ be an inner product space and $\lambda \in F$. Show that $< x, 0 >= 0 \; \forall x \in x$. [5]

(c) Let $(X, < ., . >)$ be an inner product over F. Prove that $\forall x, y \in X$,

$$| < x, y > | \leq \sqrt{< x, x >< y, y >}.$$

[8]

(d) Prove that every inner product space is a normed linear space. [8]

A3. (a) Define the term weak convergence of a sequence. [3]

(b) Let $(X, \| \, . \, \|)$ be a normed linear space. For $x, y \in X$ define $d(x, y) = \| x - y \|$. Prove that d defines a metric on X. [6]

(c) Let T be a linear operator from a normed linear space X into a normed linear space Y. Prove that the following statements are equivalent,

(i) T is continuous on X. [2]

(ii) T is continuous on 0. [3]

(iii) T is bounded. [3]

(d) Let X and Y be normed linear spaces over F. Prove that $B(X, Y)$ is a normed linear space over F. [8]

A4. (a) State and prove the projection theorem. [6]

(b) Prove that the open ball $B(x_0, \epsilon)$ is an open set. [7]

(c) Let R be the relation in the set of real numbers defined by aRb if $0 \leq a - b \leq 1$. Express R and R^{-1} as subsets of $\mathbf{R} \times \mathbf{R}$ and graph the relations. [12]

A5. (a) Define the adjoint of an operator $T : X \to Y$ where X and Y are normed linear spaces. [3]

(b) Prove that if $p > 1$ and q is such that $\frac{1}{p} + \frac{1}{q} = 1$, then $a^{\frac{1}{p}}.b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}$, where $a, b \in \mathbf{R}^+$. [7]

(c) State and prove Minkowski inequality. [7]

(d) Is the space $C[-1, 1]$ complete with respect to the metric

$$d(x, y) = \{\int_{-1}^{1} |x(t) - y(t)|^2 dt\}^{\frac{1}{2}}?$$

Justify your answer. [8]

A6. (a) Let H be a Hilbert space and M a closed subspace of H. For each $x \in H$ minus M, there is a unique element $y_0 \in M : \|x - y_0\| = inf_{y \in m} \|x - y\|$. [15]

(b) Let T_n be a sequence in $B(X, Y)$ where X is a Banach space. Assume that the sequence $(\|T_n x\|)$ is bounded for each $x \in X$ i.e $\forall x \in X \exists C_x > 0 : \|T_n x\| \leq C_x$ $\forall n$. Prove that $\exists m > 0 : \|T_n\| \leq m$. [10]

END OF QUESTION PAPER