BINDURA UNIVERSITY OF SCIENCE EDUCATION

Faculty of Science and Engineering

Department of Engineering and Physics

BACHELOR OF SCIENCE HONOURS DEGREE

Environmental Physics

JAN 2025

HPH123

Mathematics for Physicists II

Duration: Three (3) Hours

Answer any THREE questions. Each question carries 33 marks.

Clearly show ALL working

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Possibly Useful Formulae

$$f(x) = a_0 + \sum_1^{\infty} a_n cos\left(\frac{2\pi nx}{L}\right) + \sum_1^{\infty} b_n sin\left(\frac{2\pi nx}{L}\right)$$

$$a_0 = \frac{1}{L}\int_0^L f(x)dx \qquad a_n = \frac{2}{L}\int_0^L f(x)cos\left(\frac{2\pi n}{L}x\right)dx \qquad b_n = \frac{2}{L}\int_0^L f(x)sin\left(\frac{2\pi n}{L}x\right)dx$$

$$f(x) = \sum_{-\infty}^{\infty} c_n exp\left(\frac{i2\pi n}{L}x\right) \qquad c_n = \frac{1}{L}\int_0^L f(x)exp\left(\frac{-i2\pi n}{L}x\right)dx$$

$$f(x) = \int_{-\infty}^{\infty} c(k)e^{ikx}\,dk \qquad c(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,e^{-ikx}dx$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega \qquad F(\omega) = \int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}dt$$

$$cos(\alpha \pm \beta) = cos\alpha cos\beta \mp sin\alpha sin\beta$$

$$sin(\alpha \pm \beta) = cos\alpha sin\beta \pm sin\alpha cos\beta$$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{2!}f^{(n)}(a) + \cdots$$

$$\int_{-\infty}^{\infty} e^{-(x-c)^2/b^2}dx = b\sqrt{\pi}$$

$$\frac{1}{L}\int_0^L |f(x)|^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^2\,dt = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |c(k)|^2 dk \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(t)|^2\,dt = \int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega$$

Standard Laplace transforms. The transforms are valid for $s > s_0$.

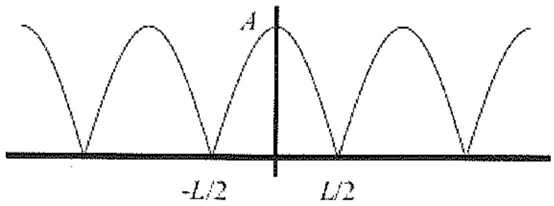
$f(t)$	$\bar{f}(s)$	s_0
c	c/s	0
ct^n	$cn!/s^{n+1}$	0
$\sin bt$	$b/(s^2+b^2)$	0
$\cos bt$	$s/(s^2+b^2)$	0
e^{at}	$1/(s-a)$	a
$t^n e^{at}$	$n!/(s-a)^{n+1}$	a
$\sinh at$	$a/(s^2-a^2)$	$\lvert a\rvert$
$\cosh at$	$s/(s^2-a^2)$	$\lvert a\rvert$
$e^{at}\sin bt$	$b/[(s-a)^2+b^2]$	a
$e^{at}\cos bt$	$(s-a)/[(s-a)^2+b^2]$	a
$t^{1/2}$	$\frac{1}{2}(\pi/s^3)^{1/2}$	0
$t^{-1/2}$	$(\pi/s)^{1/2}$	0
$\delta(t-t_0)$	e^{-st_0}	0
$H(t-t_0)=\begin{cases}1 & \text{for } t \geq t_0\\ 0 & \text{for } t < t_0\end{cases}$	e^{-st_0}/s	0

$$\mathcal{L}[f'(t)] = s\mathcal{L}[f(t)] - f(0)$$

$$\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - sf(0) - sf'(0)$$

Question 1

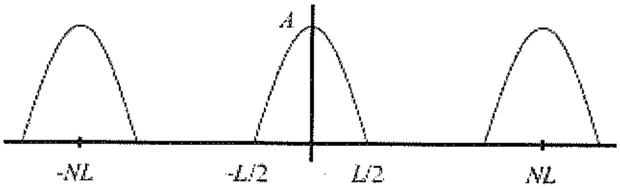
(a) Consider the periodic "cosine bump" shown.



It is defined by $f(x) = A cos(\pi x/L)$ for $-L/2 < x < L/2$, with periodicity L. Find the Fourier trigonometric series of the function.

[9]

(b) Let there now be some spacing between the bumps, as shown.



The distance between the centers of the bumps is NL, where N is a number. In terms of N, find the Fourier trigonometric series for this function. **Hint:** *It's nearly the same calculation as the one in part (a), with some extra N's in certain places.*

[9]

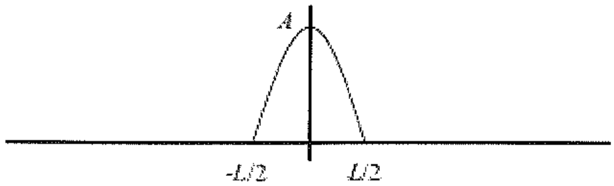
(c) Take the $N \longrightarrow \infty$ limit and then define $z \equiv n/N$, and express $f(x)$ as an integral over z.

[5]

(d) Rewrite the integral in terms of a new variable k defined by $k = 2\pi z/L$.

[4]

(e) Calculate the Fourier transform of the *single* cosine bump shown.



Verify that the $f(x) = \int_{-\infty}^{\infty} c(k)\, e^{ikx} dx$ expression agrees with your answer to part (d).

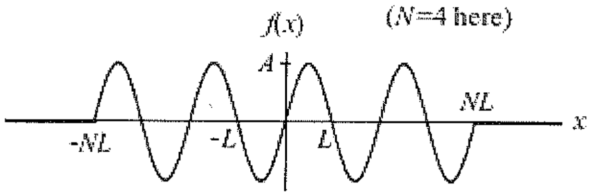
[6]

Question 2

(a) Consider the function defined by

$$S(x) = \begin{cases} A\sin\dfrac{\pi x}{L} & \text{for } -NL \le x \le NL \\ 0 & \text{for } |x| > NL \end{cases}$$

where N is an integer. The figure below shows the case where $N = 4$.



Find the Fourier transform of the function. **Hint:** *Work in terms of a general number N.*

[12]

(b) Find the value of $c(k)$ very close to $k = \pi/L$, that is, at $k = \pi/L + \varepsilon$ with $\varepsilon \longrightarrow 0$. **Hint:** *You may use a Taylor series expansion.*

[5]

(c) Show that

$$\Im\left[e^{-|x|}\right] = \frac{1}{\pi(1+k^2)}$$

[6]

(d) Prove the relation

$$i\frac{dc(k)}{dk} = \Im[xf(x)]$$

Hence find $\Im\left[xe^{-|x|}\right]$ and use Parserval's theorem to evaluate the integral

$$\int_{-\infty}^{+\infty} \frac{k^2}{\left(1+k^2\right)^4}\,dk$$

[10]

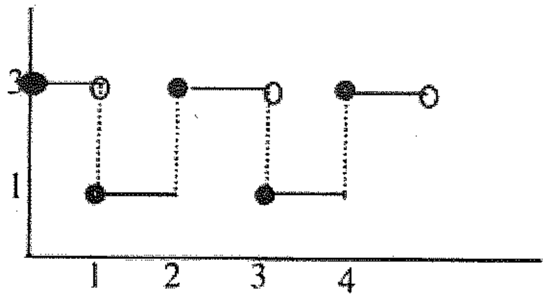
Question 3

(a) Prove that if $f(t)$ has a period $T > 0$ so that $f(t) = f(t + T)$, then

$$\mathcal{L}[f(t)] = \frac{\int_0^T e^{-st} f(t)dt}{1 - e^{-sT}}$$

[8]

(b) Find the Laplace transform of the periodic function whose graph is shown below



[5]

(c) Prove by induction that

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n \bar{f}(s)}{ds^n} \qquad n \in \{1,2,3,\cdots\}$$

Use this property to demonstrate, without evaluating any Laplace integrals explicitly, that

$$\mathcal{L}[t^{5/2}] = \frac{15\sqrt{\pi}}{8} s^{-7/2}$$

[10]

(d) Two unstable isotopes A and B and a stable isotope C have the following decay rates per atom present:

$$\begin{array}{ll} A \rightarrow B & 3\,s^{-1} \\ A \rightarrow C & 1\,s^{-1} \\ B \rightarrow C & 2\,s^{-1} \end{array}$$

Initially a quantity N_0 of A is present but there are no atoms of the other two types. Using Laplace transforms, find the amount of C present at a later time t.

[10]

Question 4

(a) By finding a suitable integrating factor, solve the equation

$$y' - y\cot x + \operatorname{cosec} x = 0$$

[8]

(b) Find, in the form of an integral, the solution of the equation

$$\alpha\frac{dy}{dt} + y = f(t)$$

for a general function $f(t)$. Find the specific solution for $f(t) = H(t)$ the Heaviside function.

[8]

(c) Find the general solution of the equation

$$\frac{dy}{dx} = \frac{4y^2}{x^2} - y^2$$

[5]

(d) Solve the simultaneous differential equations

$$\dot{x} = 2x - y$$
$$\dot{y} = y - 2x$$

where $x(0) = 8$ and $y(0) = 3$.

[12]

Question 5

(a) The function $f(t)$ satisfies the differential equation

$$\frac{d^2f}{dt^2} + 8\frac{df}{dt} + 12f = 12e^{-4t}$$

For the boundary conditions: $f(0) = 0, f'(0) = -2, f\left(ln\sqrt{2}\right) = 0$; determine whether it has solutions, and, if so, find them.

[11]

(b) Find the general solution of

$$\frac{d^3y}{dx^3} - 12\frac{dy}{dx} + 6y = 32x - 8$$

[11]

(c) The motion of a body falling in a resisting medium may be described by

$$m\frac{d^2x(t)}{dt^2} = mg - b\frac{dx(t)}{dt}$$

when the retarding force is proportional to the velocity. Using Laplace transforms, find $x(t)$ and $dx(t)/dt$ for the initial condition

$$x(0) = \left.\frac{dx}{dt}\right|_{t=0} = 0$$

What is the velocity at large t? This is the "terminal velocity."

[11]

END OF PAPER