BINDURA UNIVERSITY OF SCIENCE EDUCATION MT504

MScED Mathematics

METRIC SPACES AND TOPOLOGY AUG 2024

Time: 3 Hours

Candidates should attempt at most Four questions. Marks will be allocated as indicated.

A1. (a) (i) Define a metric on a set X. [3]

(ii) Prove that the function $d(x, y) = \|x - y\|$ for $\forall x, y \in \boldsymbol{R}^n$ is a metric on $\boldsymbol{R}^n$. [5]

(b) Let X be a metric space. Prove that if G_1 and G_2 are open in X, then $G_1 \cap G_2$ is also open in X. [6]

(c) Prove that every convergent sequence is bounded. [5]

(d) Let $A \subset \boldsymbol{R}^n, B \subset \boldsymbol{R}^n$. Prove that $A^0 \cap B^0 = (A \cap B)^0$. [6]

A2. (a) Show that for any metric space (X,d),

$$|d(z, y) - d(x, y)| \leq d(x, z) \text{ for all } x, y, z \in X.$$ [4]

(b) Show using extensionality that for all subsets of some universal set,

$$(A \cap B)^c = A^c \cup B^c.$$ [5]

(c) Let (X, d) be a complete metric space and f a contraction of X. Then, there exists a unique $x_0 \in X$: $f(x_0) = x_0$. Prove that the point x_0 is called a fixed point of f. [7]

(d) Show that $C[-1, 1]$ is not complete with respect to the metric,

$$d(x, y) = \{\int_{-1}^{1} |x(t) - y(t)|^2 \, dt\}^{\frac{1}{2}}.$$

Hint: Consider $X_n(t) = \begin{cases} 0 & -1 \leq t \leq 0 \\ nt & 0 < t < \frac{1}{n} \\ 1 & \frac{1}{n} \leq t \leq 1. \end{cases}$

[9]

A3. (a) Given that $S \subset T \subset X$, T is nowhere dense in X. Show that S is nowhere dense in X. [3]

(b) Given that $S \subset T \subset X$, S is nowhere dense in T. Show that S is nowhere dense in X. [3]

(c) Let X be a metric space. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $x_n \to x$ and $y_n \to y$, show that $d(x_n, y_n) \to d(x, y)$. [8]

(d) (i) Outline the difference between a partial order and an equivalence relation. [3]

(ii) Let A be the set on non-zero integers and let $\approx$ be the relation on $A \times A$ defined as follows:

$(a, b) \approx (c, d)$ whenever $ad = bc$. Prove that $\approx$ is an equivalence relation. [8]

A4. (a) Let T_1 and T_2 be two topologies on a non-empty set X. Show that $T_1 \cap T_2$ is also a topology on X. [8]

(b) Show that the union of two topologies is not necessarily a topology. [6]

(c) Let $X = \{a, b, c, d\}$. Determine whether or not each of the following classes of subsets of X is a topology on X,

$$T_1 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$$

$$T_2 = \{X, \emptyset, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$$

$$T_3 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c, d\}\{a, b, c, d\}\}.$$ [11]

A5. (a) Prove that the Euclidean n-space R^n is complete. [8]

(b) Let (X, d) and (Y, ρ) be metric spaces and $f: X \to y$. Prove that the following statements are equivalent,

(i) f is continuous on X.

(ii) for any open set $G \subset Y$, $f^{-1}(G)$ is open in Y.

(iii) for any closed set F in Y, $f^{-1}(F)$ is closed in X. [9]

(c) Let X be a metric space with metric d. Show that,

$$d^1 = \frac{d(x, y)}{1 + d(x, y)}$$

is a metric on X. [8]

A6. (a) (i) Define the product of a set A and set B. [3]

(ii) Show that $A \times (B \cup C) = (A \times B) \cup (A \times C)$. [7]

(b) Consider the metric space $\boldsymbol{R}$, with the usual metric and define $f: \boldsymbol{R} \to \boldsymbol{R}$ by $f(x) = (1 + x)^{\frac{1}{3}}$. Show that $f(x)$ is a contraction on [1, 2]. Using an initial guess of $x_0 = 1$, find the fixed point of $f(x)$ correct to 3 decimal places. [15]

END OF QUESTION PAPER