

METRIC SPACES AND TOPOLOGY

Time : 3 hours

AUG 2023

Candidates should attempt at most FOUR questions. Marks will be allocated as indicated

- A1.** (a) Let $\mathbf{Z}$ be the set of all integers and let m be a fixed positive integer. Two integers a and b are said to be *congruent modulo m* , symbolized by $a \equiv b \text{ modulo } m$, if $a - b$ is exactly divisible by m . Show that this is an equivalence relation and describe the equivalence sets and state the number of the distinct equivalent sets. [8]
- (b) Define a metric on a set X . [3]
- (c) Let $X = \mathbf{R}^n$ or $\mathbf{C}^n$ for $x = (x_1, x_2, x_3, \dots, x_n)$, $y = (y_1, y_2, y_3, \dots, y_n)$. Define $d : X \times X \rightarrow \mathbf{R}$ by $d(x, y) = [\sum_{i=1}^n (x_i - y_i)^2]^{\frac{1}{2}}$. Prove that (X, d) is a metric space. [8]
- (d) Let A and B be subsets of a metric space. Show that
- (i) If $A \subset B$ then $A^0 \subset B^0$. [2]
- (ii) $(A \cap B)^0 = A^0 \cap B^0$. [4]
- A2.** (a) For any sets A and B , prove that
- (i) $(A - B) \cap B = \emptyset$. [4]
- (ii) $(A \cap B)^c = A^c \cup B^c$. [4]
- (b) Let X be a metric space. Is the following statement correct? ' $F \subset X$ is closed if and only if E contains all its limit points'. Justify your reasoning. [6]
- (c) What does it mean to say that a subset S of a metric space X is sequentially compact? [2]
- (d) Let X be a metric space. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $x_n \rightarrow x$ and $y_n \rightarrow y$, show that $d(x_n, y_n) \rightarrow d(x, y)$. [9]
- A3.** (a) Consider the metric space $C[0, 1]$ with the metric d_∞ and let $f \in C[0, 1]$ be fixed, $f \neq 0$. Show that $F : C[0, 1] \rightarrow \mathbf{R} : g \rightarrow \int_0^1 f(x)g(x)dx$ is continuous for every $g \in [0, 1]$. [7]
- (b) (i) Define a nowhere dense subset S of a metric space. [3]

- (ii) Given that $S \subset T \subset X$, T nowhere dense in X , show that S is nowhere dense in X . [4]
- (iii) Given that $S \subset T \subset X$, S nowhere dense in T , show that S is nowhere dense in X . [6]
- (c) Prove that a compact set S in a metric space X is closed. [5]
- A4.** (a) Let T_1 and T_2 be two topologies on a non-empty set X . Show that $T_1 \cap T_2$ is also a topology on X . [8]
- (b) Show that the union of two topologies is not necessarily a topology. [6]
- (c) Let $x = \{a, b, c, d\}$. Determine whether or not each of the following classes of subsets of X is a topology on X .
 $T_1 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$
 $T_2 = \{X, \emptyset, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$
 $T_3 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}\}.$ [11]
- A5.** (a) Let A be the set on non-zero integers and let $\approx$ be the relation on $A \times A$ defined as follows: $(a, b) \approx (c, d)$ whenever $ad = bc$.
 (i) Prove that $\approx$ is an equivalence relation. [8]
 (ii) Let $A = \{1, 2, 3, \dots, 15\}$. Let $\approx$ be the equivalence relation on $A \times A$ defined by $(a, b) \approx (c, d)$ if $ad = bc$. Find the equivalence classes of $(3, 2)$. [2]
- (b) Show that a subset of a set of first category is also of first category. [4]
- (c) Prove that if G_n is a sequence of non-empty open and dense subsets of a complete metric space X , then $G = \bigcap_{i \in \mathbb{N}} G_i$ is dense in X . [11]
- A6.** (a) Let X be a metric space. Prove that if G_1 and G_2 are open then $G_1 \cap G_2$ is open. [4]
- (b) Let (X, d) and (Y, P) be metric spaces and $f : X \rightarrow Y$. Prove that the following statements are equivalent.
 1. f is continuous on X
 2. for any open set $G \subset Y$, $f^{-1}(G)$ is open in X .
 3. for any closed set F in Y , $f^{-1}(F)$ is closed in X . [8]
- (c) Let M be closed proper subspace of a normed linear space X . Then $\exists z \in X : \|z\| = 1$ and $\|y - z\| > 1 - \epsilon, \forall \epsilon > 0, \forall y \in M$. [6]
- (d) let (X, d) be a complete metric space and f a contraction of X . Then there exist unique $x_0 \in X : f(x_0) = x_0$. Prove that the point x_0 is called a fixed point of f . [7]

END OF QUESTION PAPER