BINDURA UNIVERSITY OF SCIENCE EDUCATION SFM412

BACHELOR OF STATISTICS AND FINANCIAL MATHEMATICS

ACTUARIAL MATHEMATICS

Time : 3 hours NOV 2024

Candidates may attempt ALL questions in Section A and at most TWO questions in Section B. Each question should start on a fresh page.

SECTION A (40 marks)

Candidates may attempt ALL questions being careful to number them A1 to A4.

A1. (a) A local bank has introduced a new student savings account that gives a student two options:

- Deposit $R2,500$ for one year and earn an effective rate of discount of 9.0913% per annum for the first six months and an effective interest rate of 10.5% per annum for the last six months or
- Deposit $R2,500$ for one year and earn 10% per annum compounded half-yearly for the entire year

Which option, if any, is best for the student from a financial perspective? Show your calculations and justify your answer fully. [5]

(b) The n -year forward rate for transactions beginning at time t and maturing at time $t+n$ is denoted by $f_{t,n}$. You are given:

- $f_{0,1} = 5.0\%$ per annum
- $f_{0,2} = 5.5\%$ per annum
- $f_{1,2} = 5.7\%$ per annum

(i) Calculate the 3-year par yield. [4]

(ii) Outline briefly the liquidity preference theory and then, explain whether (or not) the theory can be used to fully explain the term structure of spot rates above. [3]

(c) Explain what you understand by the following terms in life insurance:

(i) Select mortality

(ii) Ultimate mortality

(iii) Temporary annuity

(iv) Guaranteed annuity

[1,1,1,1]

A2. You are given that the annual force of interest is

$$\delta(t) = 0.06 + 0.004t,$$

where $t \geq 0$
Calculate the combined present value at $t = 2$ of:

(a) an annuity payable continuously at an annual rate of $40\exp(-0.03t + 0.002t^2)$ from $t = 4$ to $t = 9$; and

(b) \$300 invested at time $t = 5$. [6,4]

A3. Derive a capital gains test for an investment in a fixed interest bond. The test must be in terms of the net redemption yield on the fixed interest bond.

The following notation may be used:

- R = redemption %,
- P = price paid,
- D = annual coupons payable p times per year,
- $i^{(p)}$ = net redemption yield,
- n = term of fixed interest bond,
- t_1 = income tax, and
- t_2 = capital gains tax

[4]

A4. Assume that spot interest rates are given by the following function, where t is measured in years:

$$y_t = 0.058 + \frac{t^2}{50}$$

Calculate:

(a) the continuous-time 4-year spot rate; [2]

(b) the instantaneous forward rate, F_4; and [4]

(c) the 4-year par yield. [4]

SECTION B (60 marks)

Candidates may attempt TWO questions being careful to number them B5 to B7.

B5. (a) A decreasing 2-year term assurance pays a benefit at the end of the year of death. The sum assured is \$90,000 in the first year and \$50,000 in the second year. No benefit is payable on survival to the end of the second year.

Determine the mean and variance of the present value of the benefit for this contract issued to a life aged 62 exact.

Basis:
Mortality AM92 Ultimate
Interest rate 6.5% per annum [8]

(b) A pension fund has a liability of $R400,000$ due in ten years time. The pension fund has exactly enough funds to cover the liability based on an effective rate of interest of 8% per annum. This is also the interest rate at which current market prices are calculated and the rate earned on cash.

The pension fund wishes to hold 10% of its funds in cash, and to invest the balance in the following securities:

- A zero-coupon bond redeemable at par in 12 years time
- A fixed-coupon bond paying half-yearly coupons of 8redeemable at 110% in 16 years time

(i) Determine the amount to invest in the zero-coupon bond and the fixed-coupon bond, respectively, for the pension funds portfolio of assets and liabilities to satisfy Redingtons first two conditions for immunisation. [10]

(ii) Explain, without further calculation, whether the pension fund would be immunised against small changes in the interest rate if the quantities of stock in i. are purchased. [4]

(c) Derive the variance of the PV random variable of the following:

(i) A whole life annuity payable annually in arrears. [4]

(ii) A whole life annuity payable annually in advance. [4]

B6. (a) A fixed interest bond of nominal amount of $R1,200,000$ is issued with 8% coupons per annum, payable half-yearly in arrears. Redemption will be at 133% on any coupon date between 15 and 25 years after the date of issue. The date of redemption is at the option of the borrower.

(i) An investor who is liable for paying income tax at 40% wishes to purchase the entire nominal value of this fixed interest bond at the date of issue. The investor does not pay any capital gains tax.

Calculate the price the investor should pay to ensure a net effective yield of at least 6% per annum? [7]

(ii) Suppose that the investor pays the price calculated in (i) but the borrower redeems the fixed interest bond at a different date from the one chosen in (i).

Explain, without doing any additional calculations, what the change will be on the investors net effective yield. [2]

(b) Derive the expected present value of a benefit of $R500,000$ payable immediately on death in respect of a life currently aged 65 exactly, if death occurs within the next year.

Assume the force of mortality is constant between consecutive integer ages. [4]
Basis:
Mortality AM92 Ultimate
Interest rate 4.8% per annum [5]

(c) When pricing fixed interest bonds, the capital gains test compares $i^{(p)}$ to $(1 - t_1) \times DR$, where

- p is the frequency of the coupon payment,
- t_1 is the income tax rate,
- D is the annual coupon rate,
- R is the redemption rate, and
- i is the annual effective rate of return.

(i) Show that if $i^{(p)} > (1-t_1) \times \frac{D}{R}$, then there is a capital gain at redemption. [6]
Two situations where a capital gains test needs to be performed are:

1. for an investor who pays capital gains tax.
2. where the redemption date of the bond is variable.

(ii) Explain, for these two situations, why the capital gains test is necessary. [6]

B7. (a) Let d be the annual effective discount rate and $d^{(p)}$ be the annual nominal discount rate compounded p^{th}-ly per annum. Starting with a series representing the present value of payments of $\frac{d^{(p)}}{p}$, payable p^{th}-ly in advance, show that

$$\left(1 - \frac{d^{(p)}}{p}\right)^p = 1 - d$$

[4]

(b) A sum of $R1,500$ is accumulated at a rate of discount of 6.5% per annum convertible quarterly for nine months, then at a nominal rate of interest of 7.5% per

per annum convertible every four months for 24 months and finally at a constant force of interest of 5.25% per annum, thereafter. Calculate the accumulated amount of the investment after four years. [4]

(c) Prove that

$$d < d^{(2)} < d^{(3)} < ... < \delta < ...i^{(3)} < i^{(2)} < i$$ [5]

(d) The force of interest is given by:

$$\delta(t) = \begin{cases} 0.09 + 0.0006t^2 & 0 \leq t < 9 \\ 0.1836 & 9 \leq t < 15 \\ 0.1086 - 0.005t & 15 \leq t < 20 \\ 0.1086 & 20 \leq t \end{cases}$$

where t is measured in years.
Calculate the accumulated value at time 17 of a continuous payment stream with rate $\rho(t) = e^{0.08+0.0002t^3}$, payable for seven years starting at time 2. [8]

(e) An insurance company has liabilities of $100,000 due in four years' time and $200,000 due in 14 years' time. The company owns assets consisting of five coupon bearing bonds of $10,000 nominal each and one zero coupon bond paying $X in n years' time. The coupon bearing bonds pay 4% coupons annually in arrear for five years and are redeemable at par.
i. The current effective interest rate is 6% per annum and an attempt has been made to immunise the portfolio against small movements in interest rates. Determine X and n. [6]
ii. Without doing further calculations, state whether Redington's third condition for immunisation holds, giving a reason for your answer. [3]

END OF EXAM