BINDURA UNIVERSITY OF SCIENCE EDUCATION MT504

MScEd Mathematics

METRIC SPACES AND TOPOLOGY

Time : 3 hours AUG 2023

Candidates should attempt at most FOUR questions. Marks will be allocated as indicated .

A1. (a) Let $\mathbf{Z}$ be the set of all integers and let m be a fixed positive integer. Two integers a and b are said to be *congruent modulo m*, symbolized by $a \equiv b$ *modulo* m, if $a - b$ is exactly divisible by m. Show that this is an equivalence relation and describe the equivalence sets and state the number of the distinct equivalent sets. [8]

(b) Define a metric on a set X. [3]

(c) Let $X = \mathbf{R}^n$ or $\mathbf{C}^n$ for $x = (x_1, x_2, x_3, ..., x_n), y = (y_1, y_2, y_3, ..., y_n)$. Define $d : X \times X \longrightarrow \mathbf{R}$ by $d(x, y) = [\sum_{i=1}^{n}(x_i - y_i)^2]^{\frac{1}{2}}$. Prove that (X, d) is a metric space. [8]

(d) Let A and B be subsets of a metric space. Show that

(i) If $A \subset B$ then $A^0 \subset B^0$. [2]

(ii) $(A \cap B)^0 = A^0 \cap B^0$. [4]

A2. (a) For any sets A and B, prove that

(i) $(A - B) \cap B = \emptyset$. [4]

(ii) $(A \cap B)^c = A^c \cup B^c$. [4]

(b) Let X be a metric space. Is the following statement correct? '$F \subset X$ is closed if and only if E contains all its limit points'. Justify your reasoning. [6]

(c) What does it mean to say that a subset S of a metric space X is sequentially compact? [2]

(d) Let X be a metric space. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $x_n \to x$ and $y_n \to y$, show that $d(x_n, y_n) \to d(x, y)$. [9]

A3. (a) Consider the metric space $C[0, 1]$ with the metric d_∞ and let $f \in C[0, 1]$ be fixed, $f \neq 0$. Show that $F : C[0, 1] \longrightarrow \mathbf{R} : g \longrightarrow \int_0^1 f(x)g(x)dx$ is continuous for every $g \in [0, 1]$. [7]

(b) (i) Define a nowhere dense subset S of a metric space. [3]

(ii) Given that $S \subset T \subset X, T$ nowhere dense in X, show that S is nowhere dense in X. [4]

(iii) Given that $S \subset T \subset X, S$ nowhere dense in T, show that S is nowhere dense in X. [6]

(c) Prove that a compact set S in a metric space X is closed. [5]

A4. (a) Let T_1 and T_2 be two topologies on a non-empty set X. Show that $T_1 \cap T_2$ is also a topology on X. [8]

(b) Show that the union of two topologies is not necessarily a topology. [6]

(c) Let $x = \{a, b, c, d\}$. Determine whether or not each of the following classes of subsets of X is a topology on X.
$T_1 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$
$T_2 = \{X, \emptyset, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$
$T_3 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}\}$. [11]

A5. (a) Let A be the set on non-zero integers and let $\approx$ be the relation on $A \times A$ defined as follows: $(a, b) \approx (c, d)$ whenever $ad = bc$.

(i) Prove that $\approx$ is an equivalence relation. [8]

(ii) Let $A = \{1, 2, 3, ..., 15\}$. Let $\approx$ be the equivalence relation on $A \times A$ defined by $(a, b) \approx (c, d)$ if $ad = bc$. Find the equivalence classes of $(3, 2)$. [2]

(b) Show that a subset of a set of first category is also of first category. [4]

(c) Prove that if G_n is a sequence of non-empty open and dense subsets of a complete metric space X, then $G = \cap_{i \in N} G_i$ is dense in X. [11]

A6. (a) Let X be a metric space. Prove that if G_1 and G_2 are open then $G_1 \cap G_2$ is open. [4]

(b) Let (X, d) and (Y, P) be metric spaces and $f : X \to Y$. Prove that the following statements are equivalent.
1. f is continuous on X
2. for any open set $G \subset Y, f^{-1}(G)$ is open in X.
3. for any closed set F in $Y, f^{-1}(F)$ is closed in X. [8]

(c) Let M be closed proper subspace of a normed linear space X. Then $\exists z \in X$: $\|z\| = 1$ and $\|y - z\| > 1 - \epsilon, \forall \epsilon > 0, \forall y \in M$. [6]

(d) let (X, d) be a complete metric space and f a contraction of X. Then there exist unique $x_0 \in X : f(x_0) = x_0$. Prove that the point x_0 is called a fixed point of f. [7]

END OF QUESTION PAPER