for the following equation:

$$\frac{dx}{dt} = -1 - x - y,$$

$$\frac{dy}{dt} = 5 + 2x - y.$$ [6]

Q3. (a) Suppose we have the differential equations:

$$\frac{dx}{dt} = x(1 - x - y)$$

$$\frac{dy}{dt} = y\left(\frac{1}{2} - \frac{1}{4}x - \frac{3}{4}y\right)$$

Let x and y represent population densities of two species of bacteria competing for food supply.

(i) Examine whether there are equilibrium states that might be reached. [4]

(ii) Examine whether a periodic growth and decay will be observed. [3]

(iii) How such possibilities depend on the initial state of the two species? [8]

(b) Solve $X' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} X$. [6]

(c) Determine the stability of the equation $4r^4 + 3r^3 - 3r^2 - 6r + 10 = 0$. [4]

Q4. (a) Let $A = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$, find the Jordan form of A. [3]

(b) (i) State the Liapunov stability theorem. [3]

(ii) State when do we say a system is a strong Liapunov function. [3]

(iii) Show that the function $V(y_1, y_2) = y_1^2 + y_1^2 y_2^2 + y_2^4, (y_1, y_2) \in \mathbb{R}^2$ is a strong Liapunov function for a system:

$$\dot{x_1} = 1 - 3x_1 + 3x_1^2 + 2x_2^2 - x_1^3 - 2x_1x_2^2$$

$$\dot{x_2} = x_2 - 2x_1x_2 + x_1^2x_2 - x_2^3$$

at a fixed point. [7]

BINDURA UNIVERSITY OF SCIENCE EDUCATION **MT502**

MSc. Education: Part 1

JAN 2025

Non-Linear Differential Equations

Time: 3 hours

Candidates may attempt at most FOUR questions. Full marks can be obtained for complete solutions to FOUR questions. Each question should start on a fresh page.

Q1. (a) Which of the following equation is autonomous or non-autonomous?

(i) $X'(t) = e^t$,

(ii) $X'(t) = e^{x(t)}$,

(iii) $X'(t) = X(e^t)$. [1,1,1]

(b) Suppose function F: $R^2 \to R$ has the property that F($\tau; \varphi$)= F($-\tau; -\varphi$) for all $(\tau; \varphi) \in R^2$. Prove that if $X: R \to R$ is a solution to the differential equation $X'(t) = F(t, X(t))$, then so is Y where $Y(t) = -X(-t);\ t \in R$. [4]

(c) Find the 2×2 matrix A such that the system $X'(t) = AX(t)$ has a solution

$$X(t) = \begin{pmatrix} e^{-t}(\cos t + 2\sin t) \\ e^{-t}\cos t \end{pmatrix}, t \in R.$$ [6]

(d) Consider the I-dimensional differential equation

$$x'(t) = (x(t))^2 - 3ax(t) + 2a^2, \text{ where } a \text{ is a real constant.}$$

(i) Find the equilibrium points of the equation. [3]

(ii) The equation has qualitatively different phase portraits, depending on a. Sketch the phase portraits for the equation where $a > 0$, when $a = 0$, and when $a < 0$. [9]

Q2. (a) Solve the system $X' = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} X + \begin{pmatrix} t^{-1} \\ 2t^{-1} + 4 \end{pmatrix}$. [10]

(b) Show that the equation

$$\ddot{x} + (x^2 - \mu)\dot{x} + 2x + x^3 = 0$$

has a bifurcation point at $\mu = 0$ and is oscillatory for some $\mu > 0$. [9]

(b) Determine the critical point (x_0, y_0) and then classify its type and examine its stability by marking the transformations

$$x = x_0 + u$$
$$y = y_0 + v$$

(c) (i) State without proof the linearization theorem. [3]

(ii) Show that the system

$$\dot{x_1} = e^{x_1+x_2} - x_2$$

$$\dot{x_2} = -x_1 + x_1 x_2$$

have only one fixed point. Find the linearisation of this point. [3,3]

Q5. (a) State the Routh-Hurwitz criteria for stability of a system of ordinary differential equation. [5]

(b) Given that $X' = AX$ and $A = \begin{pmatrix} 1 & 1 & 5 \\ -4 & 0 & 2 \\ 6 & 2 & 0 \end{pmatrix}$. Determine whether the system is stable or not. [7]

(c) Consider a non-linear first logistic equation

$$\frac{dy}{dt} = f(y);\ f(y) = -r\left(1-\frac{y}{T}\right)\left(1-\frac{y}{K}\right)y, \text{ where } r > 0 \text{ and } 0 < T < K.$$

(i) Sketch $f(y)$ against y. [2]

(ii) Obtain the critical points and discuss qualitatively their stability. [4]

(iii) Sketch qualitatively the graph of $y(t)$ against t, and discuss the behavior of $y(t)$ as $t \to \infty$ when $0 < y(0) < T$ and $T < y(0) < K$. [7]

Q6. (a) What is a bifurcation? [2]

(c) Which bifurcation occurs in the following equations? Write down the first terms in Taylor expansion near the bifurcation point. Plot (qualitatively) the fixed points locations (x_*) as function of parameter value around the bifurcation point.

(i) $x' = r + x - \ln(1+x)$.

(ii) $x' = r^2 - x^2$.

(iii) $x' = x(r - e^x)$.

(iv) $x' = rx - \sinh x$ [6, 5, 6, 6]

END OF PAPER