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BINDURA UNIVERSITY OF SCIENCE EDUCATION

Faculty of Science and Engineering

Department of Engineering and Physics

BACHELOR OF SCIENCE HONOURS DEGREE

Environmental Physics

HPH113

Mathematics for Physicists I

Duration: Three (3) Hours

Answer any THREE questions. Each question carries 33 marks.

Clearly show ALL working

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Question 1

(a) Express the following complex numbers in polar form $re^{i\theta}$:

$$z_1 = i + \sqrt{3} \qquad\qquad z_2 = i - \sqrt{3}$$

Write θ as a rational number times π.

[6]

(b) Show that the two square roots of $re^{i\theta}$ are $\pm\sqrt{r}\, e^{i\,\theta/2}$. Hence find the square roots of

$$z_3 = i2 \qquad\qquad z_4 = 2 + i2\sqrt{3}$$

[8]

(c) Solve the polynomial equation with real coefficients

$$z^7 - 4z^6 + 6z^5 - 6z^4 + 6z^3 - 12z^2 + 8z + 4 = 0$$

(i) by examining the effect of setting z^3 equal to 2; and then
(ii) by factorising and using the binomial expansion of $(z+a)^4$.

Find the seven roots of the equation.

[12]

(d) Find a closed-form expression for the inverse hyperbolic function

$$y = sinh^{-1}x$$

[7]

Question 2

(a) Sum the series

$$S(x) = \frac{x^4}{3(0!)} + \frac{x^5}{4(1!)} + \frac{x^6}{5(2!)} + \cdots$$

[8]

(b) Find the interval of convergence and test the end points for the power series:

$$P(x) = 1 + 2x + 4x^2 + 8x^3 + \cdots$$

[5]

(c) Starting from the Maclaurin series for $cos\ x$, show that

$$(\cos x)^{-2} = 1 + x^2 + \frac{2x^4}{3} + \cdots$$

Deduce the first three terms in the Maclaurin series for $tan\ x$.

[6]

(d) If $f(x) = sinh^{-1}x$, and its n^{th} derivative $f^{(n)}(x)$ is written as

$$f(n) = \frac{P_n}{(1+x^2)^{n-1/2}}$$

where $P_n(x)$ is a polynomial (of order $n-1$), show that the $P_n(x)$ satisfy the recurrence relation

$$P_{n+1}(x) = \left(1+x^2\right)P_n^{/}(x) - (2n-1)xP_n(x)$$

Hence generate the coefficients necessary to express $sinh^{-1}x$ as a Maclaurin series up to terms in x^5.

[14]

Question 3

(a) Find the total differential of the function $f(x,y) = yexp(x+y)$.

[5]

(b) What is the approximate percentage change in the volume of a right circular cone if the radius of the base is changed by 3 % and the height is changed by 2 %?

[6]

(c) Show that the function $f(x,y) = x^3\ exp(-x^2 - y^2)$ has a maximum at the point $\left(\sqrt{3/2}, 0\right)$, a minimum at $\left(-\sqrt{3/2}, 0\right)$ and a stationary point at the origin whose nature cannot be determined by the above procedures.

[10]

(d) A system contains a very large number N of particles, each of which can be in any of R energy levels with a corresponding energy E_i, $i = 1, 2, \cdots, R$. The number of particles in the i^{th} level is n_i and the total energy of the system is a constant, E. Find the distribution of particles amongst the energy levels that maximises the expression

$$P = \frac{N!}{n_1!\,n_2!\cdots n_R!}$$

subject to the constraints that both the number of particles and the total energy remain constant, that is,

$$g = N - \sum_{i=1}^{R} n_i = 0 \qquad\qquad h = E - \sum_{i=1}^{R} n_i E_i = 0$$

[12]

Question 4

(a) Find the partial fraction decomposition of the functions

$$g(x) = \frac{4}{x^2 - 3x} \qquad\qquad f(x) = \frac{x-6}{x^3 - x^2 + 4x - 4}$$

[12]

(b) Prove by induction that

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

[6]

(c) Show that $Q(n) = n^4 + 2n^3 + 2n^2 + n$ is divisible by 6 for all positive integer values of n.

[10]

(d) The total relativistic energy of a particle of mass m and velocity v is

$$E = mc^2\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

Compare this expression with the classical kinetic energy, $mv^2/2$.

[5]

Question 5

(a) Use D'Alembert's ratio test to determine the behaviour of the series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^n}{n^2} \qquad\qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}n}{n^2+1}$$

Cauchy's root test may be useful in testing for convergence, especially if the n^{th} term of the series contains an n^{th} power. Use it to test the series:

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$$

[9]

(b) Use the integral test to describe the behaviour of the Riemann zeta series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

[8]

(c) By using the logarithmic series, prove that if a and b are positive and nearly equal then

$$ln\frac{a}{b} \cong \frac{2(a-b)}{a+b}$$

Show that the error in this approximation is about

$$\frac{2}{3}\left(\frac{a-b}{a+b}\right)^3$$

Hint: *Let* $a+b=2c$ and $a-b=2\delta$

[8]

(d) Find the limit of the function

$$\int_x^{\pi/2}\left(\frac{y\cos y-\sin y}{y^2}\right)dy$$

as $x \to 0$.

[8]

END OF PAPER

HPH113 Formula sheet

Binomial formula and binomial coefficients

$$(x+y)^n = \sum_{k=0}^{k=n} {}^{n}C_k \, x^{n-k} y^k \qquad {}^{n}C_k \equiv \frac{n!}{k!\,(n-k)!} \equiv \binom{n}{k} \qquad \text{for } 0 \le k \le n$$

$$^{-m}C_k = (-1)^k \frac{m(m+1)\cdots(m+k-1)}{k!} = (-1)^k \,.\, {}^{m+k-1}C_k$$

$$(1+x)^{-m/2} = \sum_{n=0}^{\infty} (-1)^n \,.\, \frac{(m+2n-2)!!}{2^n n!\,(m-2)!!} x^n \qquad m = 1,2,3\cdots$$

$$(2n)!! = 2^n n! \qquad (2n-1)!! = \frac{(2n)!}{2^n n!} \qquad 0!! = (-1)!! = 1$$

Trigonometric identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \qquad \cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$1 + \tan^2\theta = \sec^2\theta \qquad \cot^2\theta + 1 = \mathrm{cosec}^2\theta \qquad \tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2\theta = 2 \sin\theta \cos\theta \qquad \cos 2\theta = 1 - 2\sin^2\theta \qquad \tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) \qquad \sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) \qquad \cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

$$\cos(n \pm 1)\theta = \cos n\theta \cos\theta \mp \sin n\theta \sin\theta \qquad \sin(n \pm 1)\theta = \sin n\theta \cos\theta \pm \cos n\theta \sin\theta$$

Standard Maclaurin series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{for } -\infty < x < \infty,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{for } -\infty < x < \infty,$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad \text{for } -1 < x < 1,$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad \text{for } -\infty < x < \infty,$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{for } -1 < x \le 1,$$

$$(1+x)^n = 1 + nx + n(n-1)\frac{x^2}{2!} + n(n-1)(n-2)\frac{x^3}{3!} + \cdots \quad \text{for } -\infty < x < \infty.$$